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VIETA (1540–1603)

PRACTICAL ALGEBRA

SECOND COURSE

BY

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4
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COLLINS'S SECOND COURSE ALGEBRA.

W. P. I

PREFACE

The adoption of the plan of giving two courses in algebra, one early and for all students, and the other after one or two intervening years for those who specialize in science, is spreading. For the purposes of the latter a review is practically essential. This book consists of two parts, a *review* of the First Year Course, followed by *additional chapters* on more advanced topics, the whole providing the preparation for entrance to colleges of liberal arts. For a course covering the ground of both the First and Second Courses, the author's "Practical Elementary Algebra" is well adapted, though much that is in this Second Course is not in it.

Not a little has been written in this country within the past few years concerning the desirability of *unifying* the course in mathematics. A textbook for general use which aims to accomplish such an end must adapt itself to present-day conditions. Of the different branches of mathematics, the algebra easily provides the best means available of bringing together algebra, arithmetic, geometry, and applied mathematics, and this unification the present text undertakes to accomplish. In its preparation it was assumed that the pupil had taken the course usually offered in plane geometry. To be sure, the book can be used by those who have not taken geometry by simply omitting geometrical matter, whether text or exercises. In the same way, if so desired, the more purely arithmetical matter may be cut out, in part or in whole, since neither the arithmetical nor geometrical material conditions the subsequent algebraic matter.

Aside from preparation for the study of more advanced mathematics, probably the most directly practical thing learned from the study of algebra is the interpretation and use of *formulas.* Formulas introduced freely into the subject serve to vitalize it,

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since they connect it with practical problems. The attempt is made to train the student to thorough familiarity with the meaning and use of applied formulas.

The conception of functionality is developed as carefully and fully as it can be for pupils of the maturity of those who study algebra. The claims of this topic are now being strongly urged by the highest authorities.

Many of the features of the author's First Year Course, such as simplicity of presentation, minimum number of cases in topics, treatment of factoring and of graphs, constant reference to the axioms in the solution of equations, systematic testing for accuracy, translating to and from algebraic language, conservatism in the order of topics, etc., will be found likewise in this book. The inductive treatment, so prominent a feature of the First Course, is partly replaced in this by the deductive, because pupils are now far better prepared to understand its meaning.

The author is indebted to Professor E. S. Loomis of the West Side High School, Cleveland, Ohio, to Lewis Omer of the Academy at Evanston, Ill., to Miss Marie Gugle of the Central High School, Toledo, Ohio, and to Miss Theresa Moran of the High School, Stevens Point, Wis., for reading both the manuscript and the proofs and giving valuable advice. Professor H. L. Terry, State High School Inspector, Madison, Wis., read portions of the manuscript and offered helpful suggestions, as also did Professor A. L. Rhoton of Georgetown College, Ky. The following have read the proofs and given aid in other ways: President G. W. Brock, Alabama Normal College, Livingston, Ala.; Professor H. E. Cobb, Lewis Institute, Chicago; Professor Ira Condit, State Teachers' College, Cedar Falls, Iowa; Professor H. S. Hippensteel, State Normal School, Stevens Point, Wis.; and Professor T. M. Smith, Lash High School, Zanesville, Ohio.

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SUGGESTIONS TO TEACHERS

It is to be understood that the first 180 pages of this book constitute a review of the First Course. This part, containing a relatively small number of exercises, should be covered by classes much faster than the remaining portion. Different plans can be followed in passing over this part. Thus, one way is to start at the beginning and cover the whole ground rapidly, perhaps having the class do little more than read the text and model solutions and solve a few exercises here and there. Another way is to begin at factoring, or at equations, or at radicals, and then review the preceding topics as occasion requires. The numerous cross-references to earlier parts are intended to facilitate this plan. Classes usually differ greatly in preparation, even in the same school, and what would be the best course to pursue for one class might be a poor one for another.

The time allotted to advanced algebra varies greatly in different schools, and this, with varying preparation, makes it often necessary for the teacher to omit portions of the text not absolutely essential. If a course in algebraic transformations and manipulations as a preparation for college mathematics is the chief or sole aim of the study, then the more purely arithmetical and geometrical portions of the text can be omitted without injury to the continuity of the work. It is hoped, however, that these portions will be taught with the other, since they are calculated to strengthen the pupils' preparation in both algebra and mathematics in general, making the whole study much more practical. Suggestions are offered in various places for transposing the order of chapters and topics.

PRACTICAL ALGEBRA

CHAPTER I

THE FUNDAMENTAL OPERATIONS

I. THE ARABIC* AND ALGEBRAIC NOTATIONS

1. **Elementary Algebra** may be described as that branch of mathematics which employs both letters and figures in the study of numbers, and recognizes two kinds of numbers, positive and negative. Algebra is also often described as the study of the equation.

2. A **number** is one or more units (or ones). It is altogether likely that the original idea of numbers included only integers. At first a distinction was made only between one and several in a vague way. Later *counting* was introduced. Counting consists in having a reference series of objects, such as the fingers, or names to take their places, and pointing to the several objects counted, meanwhile saying the names of the standard series. The last name is the name of the number of objects counted. Later the idea of measurement was connected with the conception of numbers. Thus, to measure the length of a board, one applies his foot rule or meter stick a number of times. The number of times the measuring unit is moved, counted in the scale just described, is the number. Evidently from the standpoint of measurement number is always abstract, being a quotient of one magnitude divided by another of the same kind. Pupils are prone to fail to distinguish between abstract number itself, and the word, or figures, or letter, that denotes it.

A system for denoting numbers and making numerical calculations is called a **notation.**

*** In all probability the so-called Arabic notation was invented in India in the period from the second to the seventh centuries A.D. It was transmitted to European nations through the Arabs, and so has been known for centuries as the Arabic notation for numbers.**

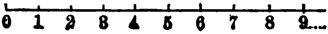
3. The Arabic Notation uses figures to denote the first nine numbers and zero, recognizes *place* value, employs absence of sign between figures to indicate addition, and is constructed on a scale of *ten*.

Thus, in 325, the 2 gets its meaning of 2 *tens* from its *place* to the left of 5, and the 3 its meaning of 3 *hundreds* from its place. Also, because juxtaposition (that is, placing figures next to each other with no sign between) means addition, 325 means $300 + 20 + 5$.

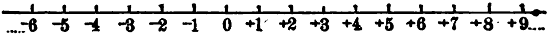
4. The Algebraic Notation uses both figures and letters to denote numbers, the letters denoting *any* numbers, but the absence of sign between letters, or between a figure and a letter, or between a figure or a letter and a parenthesis, or between two parentheses, indicates *multiplication*. For other operations special symbols are employed, as + to denote addition, −, subtraction, and so on.

Thus, $3ab + c$ denotes the number c added to the product of 3, a, and b. The algebraic notation also marks one series of numbers extending on one side of 0 with a + sign, and those on the other side of 0 extending in the opposite direction with a − sign.

Arithmetic Series:



Algebra Series:



As regards concrete applications the arithmetic series has far the wider use. The algebra series is best adapted to solving special kinds of problems which involve the presence of two kinds of numbers opposite in sense, as east and west longitude, debts and credits, numbering streets east and west or north and south, etc. But algebra *can* be used equally well in the solution of equational problems whose concrete numbers are arithmetical in character.

5. A Universal Language. The Arabic and Algebraical notations are in general use among the various civilized nations, being practically the same in different languages, though minor differences exist. Thus, in the United States and Great Britain, both ÷

and : are used to denote division, while on the continent of Europe only : is so employed.

6. Economy of Notation. Both the Arabic and the Literal notation are wonderfully abridged and compact as compared with ordinary writing in words. Thus, the number 878, which has only three figures, written out in words calls for twenty-four characters, and the formula $(a+b)^2=a^2+2ab+b^2$, translated into words, gives rise to nine times as many marks or characters.

7. Classes of Symbols in Algebra. Algebraic symbols are classified as follows:

(1) Symbols of *number*, as letters and figures.

(2) Symbols of *operations*, as $+$, $-$, $\pm$, $\times$ or $\cdot$, $\div$, $:$, $\sqrt{}$, and exponents.

(3) Symbols of *aggregation*, as ( ), [ ], { }, $\overline{\qquad}$.

(4) Symbols of *relation*, or relative size, as $=$, $\equiv$, $>$, $<$, $\not>$, $\not<$.

The angles ($>$, $<$) are called symbols of inequality, the opening being towards the larger quantity. Lines across the symbols of relation deny what the symbols unmarked signify. Thus, $\neq$ means "is not equal to"; $\not>$ means "is not greater than."

(5) Symbols of logic, as $\therefore$, meaning therefore, hence; and $\because$, meaning since, because.

(6) Symbols of continuation or omission, as 1, 2, 3, $\cdots$; see § 4.

8. The Symbols + and − are commonly considered to have two meanings:

(1) That of addition or subtraction, as $3+4=7$; $6-2=4$.

(2) That of quality, as $+5$, meaning 5 from the positive series, and -3, meaning 3 from the negative series. When the signs have the latter meaning they are usually put along with the quantity to which they are attached inside parentheses.

Thus, $(-6)-(+4)$ means 4 from the positive series is to be subtracted from 6 of the negative series.

9. The Symbols 0, 1, and ∞ are noteworthy. The first, 0, has a place in the ordinal series $\cdots -3, -2, -1, 0, +1, +2, +3, \cdots$.

It may also be regarded as belonging to the cardinal numbers. It has the following properties: $a-a=0$; $0 \times a = a \times 0 = 0$. Division by 0 has no meaning.

The symbol 1 has the following properties: $a \div a = 1$; $a \times 1 = a \div 1 = a$.

The symbol ∞ (called **infinity**) denotes an exceedingly large number, greater than any that can be named, and continually increasing.

II. DEFINITIONS

10. A **quantity** in algebra is a *number*. This number may be expressed by a single letter, or by a figure, or by a combination of figures, letters, and signs. Thus, 8, a, and $4a - 3b + 5c$ are each a quantity. "Quantity" and "algebraic expression" are used synonymously.

11. An **exponent** is a number written to the right of and higher than a quantity. When it is a positive whole number, it shows how many times the quantity is to be taken as a factor.

Thus, $4^3 = 4 \times 4 \times 4$; $a^4 = a \times a \times a \times a$.

12. A **power of a number** is the continued *product* obtained by using 1 and the number one or more times as factors.

Thus, $1 \times a \times a$, or a^2, is called the *square* of a; and $1 \times a \times a \times a$, or a^3, is called the *cube* of a; $1 \times x \times x \times x \times x$, or x^4, is called the *fourth power* of x; and so on.

13. A **root of a number** is one of its equal factors.

Thus, $\sqrt[3]{64} = 4$; $\sqrt[5]{32} = 2$. In $\sqrt[3]{64}$, 3 is called the *index* of the root. It shows into how many equal factors the number 64 is to be separated. When no index is written, 2 is understood. Thus, $\sqrt{9} = 3$. The symbol $\sqrt{\ }$ is called the radical sign.

14. A **term**, or **monomial**, is an algebraical expression, or quantity, whose parts are not separated by a + or − sign.

Thus, 3, a, $5bc$, $12a^2bc$ are each a term, while the quantity $15m^2 - 3n$ consists of two terms. But in $3(a-b) + 2(x+y)$ $3(a-b)$ is commonly regarded as a term, and $2(x+y)$ as one the parentheses quantities being treated as single letters.

15. A **polynomial** is a quantity consisting of two or more terms. A quantity consisting of two terms is a **binomial**, one of three terms is a **trinomial**, one of four terms a **quadrinomial**, one of five or more terms a **multinomial**.

By a "**polynomial in x**" is meant an expression containing different powers of x; as, $6x^5+4x^4-11x^3+x^2-7$.

16. The factors of a term are the **coefficient** and the **literal (letter) part**. Thus, in $5axy$, 5 would be regarded as the coefficient (or "with factor"). In ax^2y, 1 can be regarded as the coefficient of ax^2y, or, remembering the meaning of "coefficient," a can be regarded as the coefficient of x^2y, or even x^2 as the coefficient of ay, if so desired. When no coefficient or exponent is written, 1 is understood. Thus, $a=1a^1$.

The student should note carefully the difference between a coefficient and an exponent. Thus, $4x=x+x+x+x$, while $x^4=x\times x\times x\times x$; $4x$ means 4 times x, while x^4 is shorthand for $xxxx$. One should think of four x's multiplied together when he sees x^4.

17. **Similar terms** are those containing the same letters, the corresponding letters having the same exponents. They can differ in signs and coefficients.

Thus, $9x^2y^3$ and $-13x^2y^3$ are similar terms. Similar terms can be combined into a single term. Thus, $6a^2b+10a^2b=16a^2b$.

18. **The Degree of a Term** is the sum of the exponents of its literal factors. For example, $9m^3n^2p$ is of the sixth degree because $3+2+1=6$; but $3abx^3y$ is of only the fourth degree if we think of $3ab$ as the coefficient of the unknown factors x^3 and y.

19. **Homogeneous Quantities** are those whose terms are all of he same degree. Many expressions in algebra are homogeneous, and errors in handling such expressions can be detected by applying the test of homogeneity.

Thus, $a^3+3a^2b+3ab+b^3$ for $(a+b)^3$ is *wrong*, since all the erms should be of the third degree, and $3ab$ is of only the econd degree; this term should be $3ab^2$.

III. ORDER OF PRECEDENCE OF OPERATIONS

20. The **numerical value** of a quantity is found by assigning certain values to the letters contained in it, and simplifying the result.

Thus, the numerical value of $2a - 3m$ when $a = 5$, $m = 2$, is 4.

21. Order of Precedence of Operations. The order of performing operations in algebra is determined by certain rules.

1. *In any term* (§ 14), *symbols of aggregation being absent, raising to powers and extracting roots must be performed* **before** *multiplications and divisions.*

Thus, $2 \times 3^2 = 18$; $3x^2 = 3xx$; while $(3x)^2 = 9xx$.

2. *Symbols of aggregation being absent, multiplications and divisions must be performed* **before** *additions and subtractions.*

Thus, $3 + 4 \times 5 = 23$; $5 \times 2^2 - 2^3 \div 4 = 18$.

3. *Operations inside symbols of aggregation are performed before those outside.*

Thus, $8(3 + 4) = 8 \times 7 = 56$. $(2 \times 3)^2 = 6^2 = 36$.

EXAMPLE. If $a = 3$, $b = 2$, $c = 64$,

$$15ab^2 - 3b^3\sqrt[3]{c}$$

becomes $15 \times 3 \times 2^2 - 3 \times 2^3 \times \sqrt[3]{64}$ (By § 20)

or, $15 \times 3 \times 4 - 3 \times 8 \times 4$, or 84. (By §§ 12, 13)

a. Multiplications and divisions denoted by ×, ·, ÷ are performed in order from left to right, but when multiplications are denoted by juxtaposition, as in $4c \div 3ab$, the multiplications in dividend and divisor must be performed first. Hence, when translating this notation to that of ×, ·, ÷ parentheses must be introduced.

Thus, if $a = 12$, $b = 3$, $c = 4$.

$a \div b \times c = 12 \div 3 \times 4 = 16$; while $a \div bc = 12 \div (3 \times 4) = 1$.

22. Importance of the Rules of Precedence of Operations. If strict attention is not paid to the order in which operations are to be performed, great confusion will result.

Thus, $(2+3)\times 4^2=80$; while $2+(3\times 4)^2=146$; and $(2+3\times 4)^2=196$; and $2+3\times 4^2=50$, by the rules of § 21.

Some other rules than those just described *might* have been adopted by those who introduced the algebraic notation, but the rules of § 21 represent careful testing and are well adapted to the purpose intended. Those who study or use algebra must make their calculations conform to the laws of precedence of operations.

23. Exercise in simplifying Expressions and finding Values.

1. $3+7-6+4-2$.
2. $3+6\div 2$.
3. $4(6-3)-2$.
4. $8\times 6\div 3-12$.
5. $24\div(2\times 3)$.
6. $(5-2)(11-3\times 2)$.
7. $2+16\div 2$.
8. $2\cdot 7\cdot 5-3^2\cdot 4$.
9. $2(13+7^2-41)-15$.
10. $\sqrt{[3(19+50-42)]}+26$.

24. Exercise in translating Algebraic Expressions into Ordinary Language. To change algebraic language into ordinary words is, in many respects, like translating from one language to another, as English to German, or Latin to English. The student will find hat quantities which take many words and much circumlocution or their description in ordinary language are expressed with very ew symbols and very quickly in the algebraic notation. For this ause we can work with and reason about quantities written in he algebraic notation easily, when dealing with the same quanities in ordinary language would be almost out of the question on ccount of the complexity of the language. The great mathemaician Fourier said of the algebraic notation, "There can be no more universal or more simple language, no language more exempt rom error and obscurity."

1. Translate $4ab^2-3a^3\sqrt{c}+9c$ into ordinary language, and hen find its numerical value when $a=2$, $b=3$, $c=4$.

Translation. "From 4 times *a* times *b* times *b* take 3 times *a* times *a* imes *a* times the square root of *c*, and add 9 times *c* to the remainder."

To evaluate $4\,ab^2 - 3\,a^3\sqrt{c} + 9\,c$,
we have $4 \times 2 \times 3 \times 3 - 3 \times 2 \times 2 \times 2 \times \sqrt{4} + 9 \times 4$,
that is, $72 - 48 + 36$, or 60. *Ans.*

2. Translate $3\,x^3 + 2\,xy^2 - y\sqrt{z}$, and evaluate when $x = 3$, $y = 1$, $z = 9$.

3. Translate $3\,x(y^2 - z)$, and evaluate when $x = 5$, $y = 8$, $z = 25$.

SUGGESTION. "3 times x times the quantity which is the difference between y times y and z."

a. It is convenient to use the phrase "the quantity," or "the binomial," etc., to indicate what is included in parentheses.

Translate the following 15 exercises and evaluate them when $x = 3$, $y = 5$, $z = 4$:

4. $3\,x(y - z)$. 5. $4(xy - z)$. 6. $3\,x^2 + 2\,y^2$.
7. $9\,x^2yz - 2\,y^2$. 8. $3\,xy^2 - 2\,z\sqrt{z}$. 9. $4(x^2 + y^2 - z^2)$.
10. $3(xy^2z)^2$. 11. $4(xy^2 - x^2y)$. 12. $2(3\,xy^2 - z)^2$.
13. $(2\,x - y)^2z$. 14. $2\sqrt{y^2 - x^2}$. 15. $(x + 2\,y)(3\,y - z)$.
16. $4(2\,xy)^4z$. 17. $\sqrt{3 \cdot 2\,y^xz}$. 18. $12\,z^x + x^xy^x$.

25. Translation of Formulas into Ordinary Language.

1. If a denotes the area of a rectangle, b the length of its base, and h its height or altitude, translate the formula $a = bh$ into words.

Ans. "The number of square units in the area of a rectangle equals the product of the number of linear units in its base by the number of linear units in its altitude."

2. If c represents the total cost of n articles and p the price of one article, translate $c = np$ into words.

3. If V represents the volume of a rectangular box, l its length, w its width, and d its depth, translate $V = lwd$ into words. If S represents its surface, translate $S = 2(lw + ld + wd)$ into words.

4. If A denotes the area of a triangle, b its base, and h its altitude, translate $A = \frac{1}{2}\,bh$ into words.

5. If a represents the area of a circle, and r its radius, translate $a = \pi r^2$ into words.

6. If a represents the area of a trapezoid, b and b' its bases, h its altitude, translate $a = \frac{1}{2}(b + b')h$ into words.

7. If a represents the area of a rectangle, l its length, and b its breadth, express in words $a = bl$, $b = \frac{a}{l}$, and $l = \frac{a}{b}$ respectively.

8. If s represents the distance through which a body moves, v its uniform velocity or speed, and t the number of seconds it is in motion, what does $s = vt$ say?

9. If l denotes the length of a rectangle, and w its width, express in words $l = 2w + 3$; also $lw = 25$.

10. If c = centigrade reading, or number of degrees on centigrade thermometer, and f = reading at same time on common Fahrenheit thermometer, translate the formula, $c = \frac{5}{9}(f - 32°)$, into words.

26. Exercise in translating from Ordinary Language to that of Algebra.

1. The difference between the squares of a and b. *Ans.* $a^2 - b^2$.

2. The sum of the squares of a, b, c, and d.

3. The product of the sum of the squares and the difference of the squares of x and y.

4. The quotient of the difference between the cubes of a and x divided by the difference of their squares.

5. The difference between the product of the three factors, the square of a, b, and y, and the product of 3 and the square of x.

6. If p represents the price of eggs, n the number of dozen bought, and c the cost, state in algebraic language that the cost is the product of the price multiplied by the number of dozen bought.

27. Reading of Algebraic Expressions in Practice. While it is very important that the student should understand thoroughly the translation of expressions from algebraic language to English words, and conversely, in practice the algebraic expressions are read very much as they are written. It is customary when one comes to a polynomial in parentheses to say "the quantity" so and so. Thus, $3a+5(c-d)$ would be read "three a plus five times the quantity c minus d." It is more explicit, however, to say "the binomial," "trinomial," etc., instead of "the quantity."

The student may be interested in comparing the number of characters (letters, signs, and marks of all kinds) needed to express quantities and relations in the two notations. It will be found that as a rule it takes something like ten times as many characters in ordinary language as in the algebraic notation. Stenography, or shorthand, requires more marks than algebra.

IV. ADDITION AND SUBTRACTION

28. Addition in algebra means literally *putting together.* It differs from arithmetical addition in that both positive and negative numbers are united into one *sum,* while arithmetic adds unsigned numbers only. Thus, in algebra, debts and credits, or motions forward and backward on a straight line, can be added. In such additions positive numbers are written preceded by the sign +, or no sign, and negative numbers by −, the numbers with their signs being inclosed in parentheses. See § 8.

29. Assumed Laws Governing Addition.

I. ADDITION IS COMMUTATIVE, *that is, the order of the terms added can be changed in any way and the sum will not be altered provided each term retains its own sign.*

For example, $a+b-c=b-c+a=a-c+b$.

Evidently the sum of a set of credits and debts is the same no matter in what order they are taken.

II. ADDITION IS ASSOCIATIVE, *that is, terms can be grouped in any way by inclosing them in parentheses preceded by +, and the sum will not be changed, provided each term retains its own sign.*

Thus $(a+b)+c=a+(b+c)$.

It is clear that a set of credits and debts placed on separate sheets of paper and these arranged in any order can be *grouped* in any way with rubber bands without altering the sum.

30. Rule for Addition. The rule commonly followed in algebra is that used by the bookkeeper when he adds his debts and credits separately, and then combines the results.

31. Exercise in Addition.

1. Find the sum of \$3, − \$6, − \$11, \$22, and − \$5.

STATEMENT. $3+(-6)+(-11)+(+22)+(-5)$.

SOLUTION. $3+22=25$; $(-6)+(-11)+(-5)=-22$; $25+(-22)=3$. *Ans.*

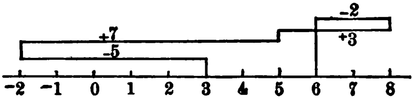
CHECK. $3+(-6)=-3$; $(-3)+(-11)=-14$; $(-14)+(+22)=+8$; $(+8)+(-5)=+3$. *Ans.*

Solve and check the following seven exercises orally:

2. 5¢ + (−3¢) + (−18¢) + (+12¢) + (−2¢) = ?

3. Sums of money deposited in a bank being marked + and other sums checked out marked −, find how much money in the bank a man has whose book shows the following items: + \$150, − \$10, − \$25, + \$30, − \$17, − \$25, + \$40.

4. Find the sum of the following lines whose several lengths are $+3$, -5, $+7$, $+3$, -2. *Ans.* $+6$.



5. Find the sum of the lines $+4$, -7, -3, $+2$, $+10$, -5, by a diagram, checking with ordinary solution.

6. If eastward motion is marked + and westward −, find the sum of the following movements of a football: +61 ft., −19 ft., +33 ft., −57 ft., +10 ft., −21 ft., +75 ft.

7. Add $9\,a^2$, $-7\,a^2$, $13\,a^2$, $-25\,a^2$, $-11\,a^2$.

8. Add $13\,mn$, $-7\,mn$, $16\,mn$, $-11\,mn$, $-2\,mn$, $-mn$.

9. Add $12\,a^2b$, $-14\,a^2b$, $-10\,a^2b$, $15\,a^2b$, $6\,a^2b$, writing the quantities for the addition in a vertical column.

10. Add $3\,a^2b$ and $3\,ab^2$. *Ans.* $3\,a^2b+3\,ab^2$. Why?

11. Add $4\,a^2x - 6\,a^2y + 9\,y^2$, $5\,a^2x - 14\,a^2y + 17\,y^2$, and $-2\,a^2x + a^2y - 11 \cdot y^3$, setting the problem down to perform the addition so that similar terms fall in the same columns. Check with $a = 2, x = 3, y = 1$.

SOLUTION

$$\begin{array}{rrrrcrrrrcr}
4\,a^2x & -\;\;6\,a^2y & +\;\;9\,y^2 & & = & 48 & -24 & +\;\;9 & & = & 33 \\
5\,a^2x & -14\,a^2y & +17\,y^2 & & = & 60 & -56 & +17 & & = & 21 \\
-2\,a^2x & +\;\;\;\;a^2y & & -11\,y^3 & = & -24 & +\;\;4 & & -11 & = & -31 \\
\hline
7\,a^2x & -19\,a^2y & +26\,y^2 & -11\,y^3 & = & 84 & -76 & +26 & -11 & = & 23
\end{array}$$

12. Find the sum of $3\,x^2y^2 - 10\,y^4$, $-x^2y^2 + 5\,y^4$, $8\,x^2y^2 - 6\,y^4$, and $4\,x^2y^2 + 2\,y^4$. Check with $x = 2, y = 1$.

13. Add $\frac{1}{3}\,x^4 - \frac{1}{5}\,x^3 - \frac{1}{2}\,x^2 + 6$, $\frac{5}{6}\,x^4 + \frac{1}{6}\,x^3 - \frac{1}{12}\,x$, $-\frac{3}{4}\,x^4 - \frac{2}{15}\,x^3 + \frac{1}{5}\,x^2 - \frac{1}{4}\,x + 1$, and $\frac{1}{12}\,x^4 - 1\frac{7}{15}\,x^3 - 2\,x^2 + 3\frac{1}{2}\,x - \frac{1}{5}$.

32. Subtraction in algebra is the process of finding a quantity (called the *difference* or *remainder*) which added to one of two given quantities (the *subtrahend*) produces the other (the *minuend*).

Thus, \$6−\$4=\$2; 4¢−(−3¢)=+7¢; −5−(+4)=−9; because in each case the algebraic sum (§ 31) of the remainder and subtrahend equals the minuend.

Subtraction in algebra can be defined in another way by referring to the algebraic series of § 4. Thus, the difference of two algebraic numbers is always the *distance* between them as they stand in the scale extending in opposite directions from 0; the difference is marked + or − according as the minuend is in the positive direction from the subtrahend or the negative direction.

As, \$7 − \$5 = \$2; 4¢ − (−3¢) = +7¢; −5 − (+4) = −9.

Evidently both definitions give the same results. The second definition is more satisfactory as an *explanation of the nature of subtraction.*

33. Two Rules for Subtraction can be given, the second of which is somewhat artificial, though generally used. Both rules always give the same result.

1. *To subtract, find a number which added algebraically to the subtrahend gives the minuend.*

2. *Conceive the sign of the subtrahend to be changed, and add the result algebraically to the minuend.*

It is highly important that the student learn to be accurate in subtraction.

34. Exercise in Subtraction. Find the remainder in the following problems, being careful to give the proper sign to the answer. The solutions should be performed mentally to the 14th. The answers can be checked by testing the result by the scale.

$$\begin{array}{llllll} \textbf{1.} & 12-(-6) & \textbf{2.} & -120-(-60) & \textbf{3.} & -16-(+25) \end{array}$$

$$\begin{array}{lrlrlr} \textbf{4.} & +24 & \textbf{5.} & -1400 & \textbf{6.} & 4\,xyz \\ & \underline{+38} & & \underline{-3200} & & \underline{15\,xyz} \end{array}$$

$$\begin{array}{lrlrlr} \textbf{7.} & 6\,a^2b & \textbf{8.} & -3\,abc & \textbf{9.} & -3\,a^2 \\ & \underline{-9\,a^2b} & & \underline{11\,abc} & & \underline{-12\,a^2} \end{array}$$

10. Find the difference in fortunes of two men, one worth \$2500, and the other \$1400 in debt, *i.e.* find the difference between \$2500 and − \$1400.

11. Find the difference in longitude between Alexandria, Egypt, 30° E. (+ 30° say), and New Orleans, 90° W. (−).

a. The first named of two quantities in a problem in subtraction is to be understood as the minuend. The answer, + 120°, shows that Alexandria, the first named, is east (+) of New Orleans, the subtrahend.

12. How many blocks apart are 2401 (each hundred means a block, and the last two figures give the house in the block) North Chestnut St. and 1401 South Chestnut St. if north is taken +?

13. What is the difference between the greatest recorded extremes of temperature at St. Paul, Minn., of + 104° and − 41°?

14. From $3\,a^2b - 2\,bc + 6$ take $2\,a^2b - 7\,bc - 3$, writing the subtrahend under the minuend with similar terms (§ 17) in the same columns to perform the subtraction. Check with $a = 2$, $b = 3$, $c = 2$.

15. From $18\,a^{2m} - 16\,a^m b^n - 25\,b^{2n}$ take $14\,a^{2m} - 24\,a^m b^n + 11\,b^{2n}$.

16. From $1\frac{1}{5}\,x^3 - 7\frac{1}{2}\,x^2 - \frac{5}{4}\,x + 11\frac{3}{5}$ take $\frac{7}{8}\,x^3 - 2\frac{1}{12}\,x^2 + 1\frac{5}{14}\,x - 7\frac{3}{8}$.

17. From $-x^4 - x^2 - x$ subtract $x^3 - 1$.

18. From the sum of 6×2^3, 2^3, and 3×2^3 take -11×2^3.

35. Addition and Subtraction in Algebra. Addition of Compound Terms. In algebra problems in subtraction, or combined additions and subtractions, can be changed into problems in addition, or can be regarded as problems in addition even when not so changed. Thus, $6 - 4$; *i.e.* $6 - (+4) = 6 + (-4)$, since each equals 2.

Similarly, $6 + 12 - 13 - 7 + 4 - 12 = ?$
becomes $6 + 12 + (-13) + (-7) + 4 + (-12) = ?$
which is now a problem in addition.

Add mentally in the following four problems, checking the result by adding seriatim, as in § 31:

1. \$12 + \$9 − \$19 + \$2 − \$13. 2. $14 - 11 - 25 - 3 + 11 + 6$.

3. Add $3(a - b)$, $5(a - b)$, and $-2(a - b)$.

4. From $12\,a(a^2 - 2\,b - c)$ take $-4\,a(a^2 - 2\,b - c)$.

5. In a merchant's cash register at the end of a day's business there were ten \$5 bills, twelve \$2 bills, twenty \$1 bills, four checks in his favor for \$12.10, \$8.25, \$10, \$25 respectively, \$18.46 in silver, a note in his favor for \$28.25, and a record on slips of paper of two notes against him, one for \$17.28 and another for \$60. His bank book showed he had \$228.68 in the bank, but during the day he had issued four checks for \$9.10, \$7.20, \$1.17, and \$5 respectively, and had received three due bills, one for \$22, the second for \$9.72, and the third for \$3.49. What will the merchant get for the *sum* if he adds all these items?

6. From $a^4 - b^4$ take $4\,a^3b - 6\,a^2b^2 + 4\,ab^3$ and from the result take $2\,a^4 - 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 - 2\,b^4$.

7. If $A = 3\,x^2 + 2\,x - 3$, $B = x^3 - 2\,x^2 + 2\,x + 1$, $C = x^2 - 6\,x + 5$, find the value of (1) $A + B + C$, (2) $A + B - C$, (3) $A - B - C$.

8. What must be added to $r^2 + s^2 + t^2$ to produce $3r^2 - s^2$?

9. Add $2(x + y) + 3(x - y)$ and $3(x + y) - 2(x - y)$.

10. Add $4(a + 2b)^2 - 4(a - m)$, $-20(a + 2b)^2 + 5(a - m)$, $12(a + 2b)^2 - 14(a - m)$, and $-5(a + 2b)^2 + 5(a - m)$.

11. Add ax, bx, and cx, with respect to x. *Ans.* $(a + b + c)x$.

SUGGESTION. These terms evidently are added just as one would add $3x$, $5x$, and $7x$.

12. Add $ax + by$, $cx + dy$, $ex + fy$, and $gx + hy$ with respect to x and y.

13. Add ax, ay and az, with respect to a.

14. Add axy and $3xy$ with respect to xy.

15. Add $a(x + y)$ and $b(x + y)$.

16. From $a(x + 2y + z)$ take $b(x + 2y + z)$.

17. Add $6 \times 2^3 - 7 \times 2^3 + 2^3 + 3 \times 2^3$.

V. SYMBOLS OF AGGREGATION

36. Removal of Symbols of Aggregation (§ 7, 3).

1. *Symbols of aggregation preceded by +, or no sign, can be removed, the signs of the terms remaining unchanged.*

Thus, $(a + b) + (c - d)$

is the same as $a + b + c - d$,

because the parentheses merely serve to group terms together, which grouping does not affect the *sum* total in any way.

2. *Symbols of aggregation preceded by − can be removed by changing the sign of every term within the symbols.*

Thus, $(6x + 2y) - (3x - 4y) - (-x + y)$

becomes $6x + 2y - 3x + 4y + x - y$, or $4x + 5y$,

since − before a parenthesis denotes that the following quantity is to be subtracted from what precedes, and this subtraction can always be effected by changing the signs of the subtrahend and adding (annexing) them to preceding quantity.

a. The + and − signs *before* a symbol *disappear* when the symbol is removed, since they denote an operation, and the operation is performed when the symbol is removed.

Care is needed in handling a vinculum quantity lest the student regard the sign immediately before the first term as belonging to this term instead of to the vinculum quantity. Thus, in $-\overline{2a+b}$, the − belongs to the whole quantity, $2a+b$, and + is understood before $2a$ inside the vinculum. In $-\overline{-2a+b}$ we see the sign − written before $2a$.

Remove symbols of aggregation and simplify, or add, in the following. After solving with pencil, solve mentally.

3. $(1-2x+3x^2)+(3-x^2+2x)$.

4. $(2x^2-3a^2x^2+9)-(x^2-5a^2x^2-3)$.

5. $(x^3-11xyz+3a)-(6xyz+7-2a-5xyz)$.

6. $-(5cz+4x^2y^3-8m)-(-cz+2x^2y^3-4cz)$.

7. $\overline{a-b-c}+\overline{b+c-d}-\overline{e-d-f}-\overline{-f+g-e}$.

8. $(3x^2+3y^2)-\{(x^2+2xy+y^2)-(-x^2+2xy+y^2)\}$.

a. We have here symbols of aggregation within other symbols of aggregation. One way for the beginner to remove such symbols is to remove only the innermost first, letting the other parts of the problem remain unchanged, then remove the next innermost, and so on.

However, all such symbols can be removed in one operation, beginning with the outermost, by noting the number of minus signs which affect any term, and making such term + if this number is even, and − if it is odd.

9. $a-\{b-(c-d)\}$. **10.** $(a+2b)-[3b-(6a-6b)]$.

11. $7a-\{3a-[4a-(5a-2a)]\}$.

12. $1-\{2-(1-x+x^2)\}$.

13. $5a-[a+5x-\{a-x-\overline{3a-2x}\}]$.

14. $2x-[3y-\{4x-(5y-6x)\}]$.

37. Inserting Quantities within Symbols of Aggregation. From the preceding articles it follows, by reversing the operation, that

1. *Any terms can be inserted within a symbol of aggregation preceded by +, or no sign, without altering their signs.*

2. *Any terms can be inserted within a symbol of aggregation preceded by − by changing the sign before each term so inserted.*

3. Write $a^2+2\,ab+b^2-c^2+2\,cd-d^2$ as a binomial, placing the first three terms in one parenthesis and the last three in another, the latter preceded by −.

4. Write $12\,ax+12\,ay+4\,by-12\,az-15\,cx+2\,bz+6\,cy-3\,cz$ so that the terms which contain x may appear as one quantity, the terms which contain y as another, and the terms which contain z as another, the last subtracted from the sum of the other two.

5. A man who owns a mill worth \$$a$ has personal property worth \$$b$ and has \$$d$ in the bank. But he owes \$$e$ for his engine and owes his millwright \$$f$. However, the millwright is indebted to him for \$$g$ worth of flour. How can what he is worth be represented so as to keep the mill separate from the rest, the accounts separate from the other items, and the money due from the millwright separate?

SOLUTION. $a+\{b+d-(e+\overline{f-g})\}$.

6. Remove the symbols from the answer to the preceding exercise.

VI. MULTIPLICATION

38. Multiplication, when the multiplier is a whole number, is the process of taking a quantity (the *multiplicand*) as many times as there are units in the *multiplier*. For other kinds of multiplication, as when the multiplier is a fraction, like $\frac{2}{3}$, or an irrational, like $\sqrt{5}$, this definition fails to have meaning. Such multiplications are extensions of the idea of multiplication by integral multipliers.

39. Laws assumed to hold in Multiplication.

I. MULTIPLICATION IS COMMUTATIVE.

Thus, $ab=ba$; $abc=acb=cba$.

a. Since $a\times b\times\frac{1}{c}\times d\times\frac{1}{e}$ must be commutative, by commutative law, then $a\times b\div c\times d\div e$, the same thing, is commutative.

Each letter, however, must not be separated from the operation symbol before it in changing the order of the operations.

Thus, $a\times b\div c\times d\div e=a\div c\times b\div e\times d=d\div c\times a\div e\times b=$ etc.

Because the operation symbols stand to the *left* of their quantities, *it is understood that the operations of multiplication and division when denoted by ×, :, ÷, ·, will be performed in any given expression in the order they come from left to right.* (See § 21, *a*).

For example, $12 \div 4 \times 3 \times 6 \div 2 = 27$; $12 \times 3 \div 4 \div 2 \times 6 = 27$.

II. MULTIPLICATION IS ASSOCIATIVE.

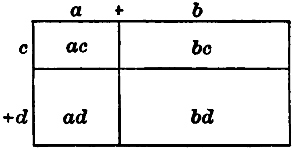
For example, $(ab)c = a(bc)$.

III. MULTIPLICATION IS DISTRIBUTIVE. *In other words, in every multiplication problem in both algebra and arithmetic, every term of the multiplicand must be multiplied by every term of the multiplier and the partial products added.*

It is well known to the pupil that this is always done in arithmetic in the multiplication of two such numbers as 2456 and 3985.

ARITHMETICAL EX. ALGEBRAICAL EX. GEOMETRICAL ANALOGUE

$$\begin{array}{r} 23 \\ \underline{25} \\ 115 \\ \underline{46} \\ 575 \end{array} \textit{ Ans.} \qquad \begin{array}{r} a+b \\ \underline{c+d} \\ ac+bc \\ \underline{+ad+bd} \\ ac+bc+ad+bd \end{array} \textit{ Ans.}$$



40. The Signs in Multiplication.

1. When the multiplier is positive, the product has the same sign as the multiplicand. Thus, three credits of + $5 each give a credit of + $15, while four debts of − $7 each give a debt of − $28.

This suggests that when the multiplier is negative, the product should have the sign opposite to that of the multiplicand. We will now test this natural assumption to see if it is in agreement with our rules for addition and subtraction and the distributive law (§ 39) which we assumed to hold true for multiplication.

Thus,
$$\begin{array}{rcr} 9-4 & = & 5 \\ \underline{7-3} & = & \underline{\times 4} \\ 63-28 & & 20 \\ \underline{-27+12} & & \\ 63-28-27+12 & = & 20 \end{array}$$

EXPLANATION. When 9 is multiplied by −3, the product, by the rule just assumed, ought to have a sign opposite to that of +9, and in multiplying −4 by −3, the product ought to have a sign opposite to that of −4, or be +12.

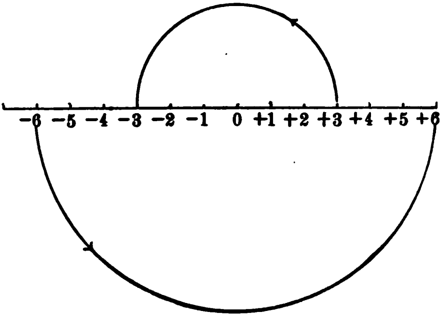
Testing our result, which we see by multiplying 5 and 4 together must equal $+20$, we find that, by our rule for addition,

$$63 - 28 - 27 + 12 = +20.$$

Hence we see that multiplication based on the distributive law and the two laws for signs just laid down agrees with addition. We have then this rule:

2. *In multiplication like signs in the two factors give a positive product and unlike signs give a negative product.*

3. How a positive multiplier gives the product the same sign as the multiplicand and a negative multiplier the opposite sign can be explained geometrically as follows: A positive multiplier simply repeats the number in the multiplicand. Hence, if the multiplicand is positive the product is positive, and if negative, the product is negative. But a negative multiplier *reverses the direction* of the multiplicand. Thus, $-1 \times +3 = -3$, that is, -1 as multiplier reverses the direction of $+3$, giving -3. Similarly in -2×-3, the multiplier -2 first *doubles* the multiplicand -3, and then reverses its direction, giving $+6$. Thus, a negative number × a negative number gives a positive product.



4. Considering a larger number of factors, we see readily that the sign of a product consisting of any number of positive factors and an *even* number of negative factors, as

$$a \times b \times c \times -d \times -e \times -f \times -g,$$

is positive, since by the associative law (§ 39), every pair of negative factors can be associated together, giving a single positive factor, so that all the factors become positive.

Thus, $a \times b \times c \times (-d \times -e) \times (-f \times -g)$

becomes $a \times b \times c \times de \times fg = +abcdefg.$

If a product consists, on the other hand, of any number of positive factors and an odd number of negative factors, the product is negative, since the last negative factor, which may be taken as coefficient (§ 39, I), will make the whole product negative.

Thus, $a \times b \times -c \times -d \times -e = -e \times a \times b \times (cd) = -abcde$.

Hence we have this rule for sign of product in multiplication:

5. *If there is an odd number of negative factors, the product is negative; otherwise it is positive.*

41. Exponents in Multiplication. The product of $2x^2$ and $3x^3$ is $6x^5$, since $2x^2 \times 3x^3 = 2xx \times 3xxx = 6x^5$, by the definition of an exponent, § 11, and by § 39, I. Similarly, $3x^4 \times 5x^6 = 15x^{10}$.

We see from these examples that the *coefficient* of the product is equal to the product of the coefficients of the factors, and the *exponents* of a letter in the factors are *added* to get the exponent of this letter in the product.

42. General Proof of the Exponent Law. Let a^m and a^n be the two factors of a product, both being integral powers of the same quantity, a.

Then, $a^m = a \times a \times a \times a \times a \cdots$ to m factors, by definition (§ 12),

and $a^n = a \times a \times a \times a \times a \cdots$ to n factors, by definition.

Hence,

$$a^m \times a^n = (a \times a \cdots \text{to } m \text{ factors})(a \times a \cdots \text{to } n \text{ factors})$$
$$= a \times a \times a \times a \times a \times a \times a \cdots \text{to } m+n \text{ factors}.$$

$\therefore\ a^m \times a^n = a^{m+n}$. (Since a^{m+n} means $a \times a \times a \cdots$ to $m+n$ factors.)

Therefore, we have proved generally that,

In multiplication the exponent of a factor in the product is equal to the sum of the exponents of this factor in the several quantities multiplied together.

43. Exercise in Multiplication. Perform the multiplication in the first 14 exercises mentally.

1. $+3 \times -4$¢. 2. -7×-8¢. 3. -6×19.

4. $3\,a^2b \times 2\,ab \times 5\,a^2bc$. *Ans.* $30\,a^5b^3c$.

5. $3\,x^2y \times -2\,xy$.

6. $4\,m^3n \times -7\,n^2$.

7. $(-2\,b + 3\,c) \times -8\,c$.

8. $4 \times 2^3 \times 3^2 \times -5^2 \times 2^2$.

9. $6 \cdot 5^2 \times 2 \times 5$.

10. $3^2 \cdot 5^2 \cdot 10 \times 2^2$.

11. Multiply $3\,y^2 + y - 1$ by $-24\,y^3$; $2 \times (-2\,x + 3\,y - 2\,z)$.

12. Multiply $5\,x + 4\,y$ by $3\,x - 2\,y$ (§ 39, III).

Ans. $15\,x^2 + 2\,xy - 8\,y^2$.

13. $(2\,ac^2 - 3\,by)(2\,c^3 - 3\,y^2)$; $(x^2 + y)(x + y^2)$.

14. $2\,a^m \times -3\,a^n$; $5\,x^m \times -2\,x^{2m}$; $2\,a^mb^n \times -2\,a^nb^m$.

15. $(5\,x^3 - 7\,x - 8)(x^2 - 2\,x + 1)$.

16. $(a^m + b^n + c^p)(a^m + 2\,b^n - 3\,c^p)$.

17. $(ax^2 + bx + c)(dx^2 - ex - f)$. Check with $a = 1$, $b = 2$, $c = 1$, $d = 3$, $e = 2$, $f = 5$, $x = 2$.

Solution.

$$\begin{array}{lr}
\quad ax^2 + bx \quad + c & = \quad 9 \\
\underline{\quad dx^2 - ex \quad - f} & \underline{= \times 3} \\
adx^4 + bdx^3 + cdx^2 & 27 \\
\qquad\quad - aex^3 - bex^2 - cex & \\
\underline{\qquad\qquad\qquad - afx^2 - bfx - cf} & \\
\multicolumn{2}{l}{adx^4 + (bd - ae)\,x^3 + (cd - be - af)x^2 - (ce + bf)x - cf = 27.}
\end{array}$$

18. $(mx^2 - 3\,nx + 4\,p)(px^2 - 5\,qx + 2\,r)$.

19. Show that

$$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x + (ab + ac + bc)x + abc.$$

20. $-3^2 \times 4$.

a. This problem may prove confusing to the pupil. Notice 3^2 as in $-a^2 \times 4$ has the coefficient -1 understood. Then $-1 \times 3^2 \times 4$ equals what by law of precedence of operations? If an algebraist desires to square -3 he writes it $(-3)^2$.

21. $6 \times (a \times b \times c)$.

22. $6 \times (a + b + c)$.

23. $4 \times (5 \times 3 \times 2)$.

24. $4 \times (5 + 3 + 2)$.

b. Evidently to multiply the *product* of several factors by a monomial multiplier, we must multiply only *one* of the factors by this multiplier, leaving the others unchanged; to multiply the *sum* of several terms by a monomial multiplier, we multiply each term by the multiplier, adding the products.

Perform the multiplications in the following exercises mentally without copying the problems:

25. $(x+3)(x+5)$.

26. $(2x+3)(3x+4)$.

27. If $3m-2$ feet is the width and $4m+3$ feet is the length of a rectangle, what is its area?

28. If $7m-3$ inches is the base and $2m+4$ inches is the altitude of a parallelogram, what is its area?

29. If $3y+7$ feet and $y+3$ feet are the parallel sides and $3y$ is the altitude of a trapezoid, what is its area?

30. At $2n-13$ cents apiece what will $5n-2$ oranges cost?

44. Powers and Roots of Monomials. Solve orally.

1. $(2a)^2$.

2. $(3m^3)^2$.

3. $(5ac)^2$.

4. $(-3a^2b)^2$.

5. $(\frac{1}{2}m^3)^2$.

6. $(22a^4)^2$.

7. $(-3a^mb^n)^2$.

8. $(a^2)^m$.

9. $(x^3)^3$.

10. $(-3x^3)^3$.

11. $(3a^n)^3$.

12. $(-6a^2b^n)^3$.

13. $(-2)^4$.

14. $(-2a^2)^5$.

15. $(a^m)^6$.

16. $(a^m)^n$.

17. $\sqrt{25a^2}=?$ *Ans.* $\pm 5a$. Why? Prove.

18. $\sqrt{16a^4}$.

19. $\sqrt{4a^{2m}}$.

20. $\sqrt[3]{-8a^3}$.

21. $\sqrt[3]{27m^9}$.

22. $\sqrt{64a^6b^4}$.

23. $\sqrt[n]{a^{2n}b^{3n}}$.

45. Arrangement of the Terms in a Polynomial and of the Factors in a Term. In the operations of multiplication, division, extraction of roots, etc., it is almost always very desirable to keep all quantities arranged in regular order, according to the ascending or descending powers of some *leading letter*, as

$$ax^3+bx^2-cx-d;\ 1+2m-m^3+3m^5.$$

The first of these expressions is arranged according to the *descending* powers of x, and the second according to the *ascending* powers of m.

In a single term (§ 14), it is the rule to arrange the letters *alphabetically*, as $5ab^3mx^2y$, though this is not essential.

The quantities heretofore given in addition, subtraction, and multiplication were quite regularly arranged according to these rules.

VII. DIVISION

46. Division is the inverse of multiplication, as subtraction is the inverse of addition. Thus division is the process of finding one factor when the product and the other factor are given.

Division bears much the same relation to subtraction that multiplication bears to addition. For integral numbers as multipliers, multiplication is often defined as a short method of addition when the numbers to be added are equal. Then division is the process of finding how many times the divisor can be subtracted from the dividend and what remainder will be left over.

47. Signs in Division. The rule for signs in division is quite like that for multiplication, viz.;

Like signs in the dividend and divisor give + for the sign of the quotient, and unlike signs give −.

This rule follows logically from the definition of division just given and the rule for signs in multiplication (§ 40, 2). Thus,

$+5 \times +6 = +30$, whence $+30 \div +5 = +6$;

or a + number ÷ a + number gives a + quotient.

$-5 \times -6 = +30$, whence $+30 \div -5 = -6$;

or a + number ÷ a − number gives a − quotient.

$+5 \times -6 = -30$, whence $-30 \div +5 = -6$;

or a − number ÷ a + number gives a − quotient.

$-5 \times +6 = -30$, whence $-30 \div -5 = +6$;

or a − number ÷ a − number gives a + quotient.

48. Exponents in Division.

The quotient of $10x^5 \div 2x^2 = 5x^3$, since $2x^2 \times 5x^3 = 10x^5$.

Similarly, $20x^7 \div 5x^3 = 4x^4$.

Thus, we see in these cases that the *coefficient* of the quotient equals the coefficient of the dividend divided by the coefficient of the divisor, and that the exponent of a letter in a quotient is equal to the exponent of the same letter in the dividend diminished by its exponent in the divisor.

To prove this generally, let a^p be the dividend and a^q the divisor, p and q being integral numbers.

Then $a^p \div a^q = a^{p-q}$, since $a^{p-q} \times a^q = a^p$ (by § 42). Hence

In division the exponent of a factor in the quotient equals the exponent of the same factor in the dividend diminished by its exponent in the divisor.

49. Exercise in Division. Perform the division in the first 12 exercises mentally.

1. $14\,a^2bc^3 \div 7\,ac$.
2. $-6\,a^4b^2 \div 2\,a^2b$.
3. $-24\,r^5s^2 \div -4\,rs$.
4. $-x^m \div x^2$.
5. $(a+b)^m \div (a+b)^n$.
6. $a^nb^2 \div ab$.
7. $14\,a^mc^p \div -2\,a^2c^3$.
8. $-16\,a^xy^b \div -8\,a^b$.
9. $(12 \cdot 3^5) \div (2 \cdot 3^2)$.
10. $(3\,xy + 2\,x^2y) \div xy$.
11. $6(a+x)^{y+2} \div 3(a+x)^{z-2}$.
12. $(6\,a^mb^4 + 3\,a^{m+1}b^3 - 4\,a^{m-2}b^2 - 12\,a^mb) \div 2\,ab$.
13. $(15\,x^2 + 16\,xy - 15\,y^2) \div (5\,x - 3\,y)$.

SOLUTION.

$$\begin{array}{r|l} 15\,x^2 + 16\,xy - 15\,y^2 & 5\,x - 3\,y \\ \cline{2-2} \underline{15\,x^2 - 9\,xy} & 3\,x + 5\,y \ \textit{Ans.} \\ 25\,xy - 15\,y^2 & \\ \underline{25\,xy - 15\,y^2} & \end{array}$$

14. Divide $6\,x^2 + 5\,xy - 4\,y^2$ by $3\,x + 4\,y$.
15. $(4\,x^5 + 4\,x - x^3) \div (3\,x + 2\,x^2 + 2)$.
16. $(m^3 - n^3) \div (m - n)$; $(a^5 - 243) \div (a - 3)$.
17. $(16\,x^3 - 29 + 27\,x - 46\,x^2) \div (8\,x - 3)$.

a. When any division ends with a remainder, test the work by multiplying divisor and quotient together and adding the remainder to the product.

18. $(21\,a^3 - 4\,a^2 - 12 - 42\,a) \div (4\,a + 2 - 3\,a^2)$.
19. Divide $1 + 2\,x$ by $1 - 3\,x$, getting 5 terms in the quotient. Prove answer.
20. $(x^{m+n} + x^ny^n + x^my^m + y^{m+n}) \div (x^n + y^m)$.

21. Find in the quickest way the numerical value of $x^4 + x^3 - 4x^2 + 5x - 3$ divided by $x^2 + 2x - 3$, when $x = 3$.

22. Find in the quickest way the numerical value of $(x^4 - 6x^3y + 9x^2y^2 - 4y^4) \div (x^2 - 3xy + 2y^2)$, when $x = 3$, $y = -1$.

23. Find in the quickest way $(x^4 - 6xy - 9x^2 - y^2) \div (x^2 + 3x + y)$, when $x = 4$, $y = 2$.

24. The area of a rectangle is $24a^2bc$ square inches and its base is $8bc$ inches. Find its altitude.

25. The cost of some articles was $36(m+n)(c-d)$ dollars and the price was $2(c-d)$ dollars. Find the number of articles.

26. The base of a triangle is $3abc$ feet and its area $24a^3bc^2$ square feet. Find its altitude.

50. Division by Detached Coefficients.

1. Divide $x^5 - 28x^3 + 8x^2 - 35x + 21$ by $x^3 - 5x^2 - 7$.

Solution. We write the coefficients only in regular order, filling in missing powers of x with 0 coefficients.

$$
\begin{array}{l|l}
1 + 0 - 28 + 8 - 35 + 21 & \underline{1 - 5 + 0 - 7} \\
\underline{1 - 5 + 0 - 7} & \underline{1 + 5 - 3} \\
\qquad 5 - 28 + 15 - 35 & \text{or, } x^2 + 5x - 3 \quad \textit{Ans.} \\
\qquad \underline{5 - 25 + 0 - 35} & \\
\qquad\quad - 3 + 15 + 0 + 21 & \\
\qquad\quad \underline{- 3 + 15 + 0 + 21} &
\end{array}
$$

If the student does not understand this solution, if he will perform the division as in the preceding article, supplying always missing powers of x with 0 coefficients in every case, and will compare such solution with that just given, he will understand the process.

2. Divide $x^3 + 4x^2 + 7x + 6$ by $x^2 + 2x + 3$.

3. $(a^4 - 8a^2x^2 + 16x^4) \div (a^2 + 4ax + 4x^2)$.

Suggestion. Write the dividend $1 + 0 - 8 + 0 + 16$, thinking of it as standing for $a^4 + 0a^3x - 8a^2x^2 + 0ax^3 + 16x^4$.

4 $(15x^4 + 7x + 7x^3 + 15x^2 + 4) \div (3x^2 + 2x + 1)$.

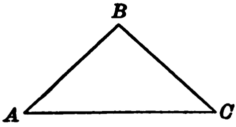
5. $(x^4 + x^2y^2 + y^4) \div (x^2 - xy + y^2)$.

6. $(1 + 5x^3 - 6x^4) \div (1 - x + 3x^2)$.

VIII. THEOREMS OF GEOMETRY

51. Theorems from Geometry. *For convenience of reference* we give here a *list* of the theorems of geometry which will be used in the remaining parts of the book. If a review of geometry is desired, the pupil should be asked to prepare anew the geometrical proofs of these propositions before he uses them in the solution of algebraical problems. If a review of geometry is not desired, the geometrical exercises themselves, of course, may be omitted.

1. *The angles opposite the equal sides of an isosceles triangle are equal, and, conversely, in any triangle, the sides opposite two equal angles are equal.*



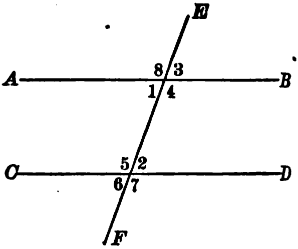
Thus, if $AB = BC$, then angle $C =$ angle A, and if angle $C =$ angle A, then $AB = BC$.

2. *An equilateral triangle is equiangular, and conversely.*

3. *If two straight lines intersect, the vertical angles formed are equal.*

4. *If two parallel lines are cut by a transversal, alternate interior angles are equal, and corresponding angles are equal, and conversely.*

Thus, if AB is parallel to CD, $\angle 1 = \angle 2$, $\angle 4 = \angle 5$, $\angle 3 = \angle 2$, etc.



5. *The opposite sides and angles of a parallelogram are equal, and the diagonals bisect each other.*

6. *The sum of the interior angles of a triangle equals two right angles.*

7. *The sum of the interior angles of an n-gon (n-sided polygon) equals $2n - 4$ right angles.*

8. *A radius perpendicular to a chord bisects it and also the subtended arc, and conversely.*

9. *Parallel chords intercept equal arcs on the circumference.*

10. *An inscribed angle, or an angle formed by a tangent and a chord, is measured by one half of the intercepted arc.*

11. *An angle formed by two chords intersecting within a circle is measured by one half of the sum of the intercepted arcs, and one formed by two secants meeting without a circle, or a secant and a tangent, or two tangents, is measured by one half of the difference of the intercepted arcs.*

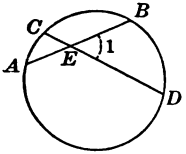


FIG. 1

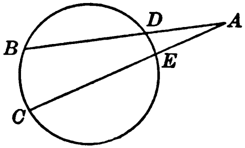


FIG. 2

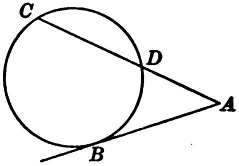
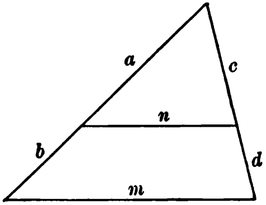


FIG. 3

Thus, $\angle 1$ (fig. 1) is measured by $\frac{1}{2}(AC+BD)$. Angle A (fig. 2) is measured by $\frac{1}{2}(BC-DE)$. Angle A (fig. 3) is measured by $\frac{1}{2}(BC-BD)$.

12. *If a line is drawn parallel to the base of a triangle, it divides the other two sides into segments which are proportional to each other and to the whole sides, and conversely.*



Thus, if n is parallel to m,

$$\frac{a}{b}=\frac{c}{d};\quad \frac{a}{c}=\frac{b}{d};\quad \frac{a+b}{b}=\frac{c+d}{d};$$

$$\frac{a+b}{a}=\frac{c+d}{c}.$$

13. *The bisector of an angle of a triangle, whether interior or exterior, divides the opposite side into segments which are proportional to the other two sides.*

Thus, if ABC is a triangle and CD and CD' are the bisectors of the interior and exterior angle C, we have

$$\frac{AD}{DB}=\frac{AC}{CB};\quad \frac{AD'}{D'B}=\frac{AC}{CB}.$$

14. *Two triangles are similar (that is, their corresponding angles are equal and their corresponding sides are proportional):*

(1) If they are mutually equiangular, or

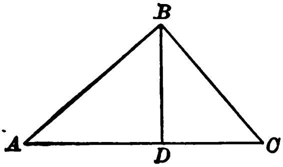
(*2*) *If an angle of one is equal to an angle of the other and the sides about these angles are proportional, or*

(*3*) *If their corresponding sides are proportional.*

15. *Two right triangles are similar if they have merely one acute angle of one triangle equal to an acute angle of the other.*

16. *If a perpendicular is dropped from the vertex of the right angle on the hypotenuse of a right-angled triangle:* (*1*) *the perpendicular is a mean proportional between the segments of the hypotenuse;* (*2*) *each leg is a mean proportional between the whole hypotenuse and its adjacent segment. Thus,*

$$\frac{AD}{DB}=\frac{DB}{DC}, \text{ and } \frac{AC}{AB}=\frac{AB}{AD}.$$



17. *If two chords intersect in a circle, their segments are reciprocally proportional. Or* (*fig. 1 under 11 above*),

$$\frac{AE}{CE}=\frac{DE}{BE}.$$

18. *If two secants intersect without a circle, they are reciprocally proportional to their external segments. Or* (*fig. 2 under 11*),

$$\frac{AB}{AC}=\frac{AE}{AD}.$$

19. *If a tangent and a secant intersect, the tangent is a mean proportional between the whole secant and its external segment. Thus* (*fig. 3, under 11*), $\frac{AC}{AB}=\frac{AB}{AD}.$

20. *Pythagorean Prop.*—*The square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides.*

21. *One side of a regular hexagon inscribed in a circle is equal to the radius.*

22. *One side of a regular decagon inscribed in a circle of radius r is x in the proportion* $\frac{r}{x}=\frac{x}{r-x}.$

23. *Theorems about the* **areas** *of a rectangle, parallelogram, triangle, trapezoid, circle, etc.*

IX. FORMULAS

52. Formulas are truths or rules expressed in algebraical symbols. They are largely used in pure and applied science.

1. If $S =$ number of right angles in the sum of the interior angles of a polygon of n sides, by the theorem of the preceding article we have the formula, $S = 2n - 4$.

With this formula find the sum of the interior angles of polygons of 3, 4, 7, 11, 25 sides respectively. Change the result in each case to degrees.

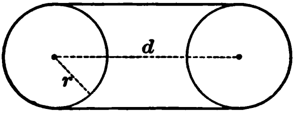
2. If $A =$ the number of right angles in one angle of an equiangular polygon of n sides, by the preceding exercise we have the formula, $A = \frac{2n-4}{n}$.

Find from this formula one angle of equiangular polygons of 3, 4, 5, 6, 8, 10, 12, 30, 100 sides respectively. Change the result in each case to degrees.

3. If v is the number of degrees in the vertical angle of an isosceles triangle, make a formula giving the value of b, one of the base angles of the triangle, in degrees. Find b when $v = 25°$.

4. Show that the formula for the length l of a belt passing round two equal pulleys whose radii are r ft. and the distance between whose centers is d ft., is $l = 2\pi r + 2d$.

Find l when $r = \frac{1}{2}$ and $d = 3\frac{1}{2}$.



5. Show that the formula for the area a of a sector of a circle whose angle at the center has d degrees in it, r being its radius, is $a = \frac{d}{360}\pi r^2$.

Find a when $r = 7$ in. and $d = 40$.

6. Make a formula for the area S of a regular polygon of n sides one of whose sides is s and whose apothem is a. Find S when $n = 9$, $s = 1.5$, $a = 20.7$ in.

NOTE. Pupils should search for practical formulas and bring them to class, showing their use.

X. FUNCTIONS

53. A function of a quantity is any algebraical expression which contains this quantity, and which therefore *depends* for its value on the value of the quantity.

Thus, $\frac{3n+5}{2}$ is a function of n. If $n=2$, $\frac{3n+5}{2}=5\frac{1}{2}$; if $n=3$, the function equals 7; and so on for other values of n.

1. The length of a rectangle is x ft. and its width is $x-7$ ft. Find its area. *Ans.* x^2-7x sq. ft.

2. A is x years old now; how old will he be in $x+1$ years from now?

3. What number must be added to x to make y?

4. Two of the angles of a triangle have n and $2n+1$ degrees in them respectively. What function of n expresses the size of the third angle?

5. What must a be increased by to give $4-3a$?

6. The length, breadth, and thickness of a rectangular solid are $4y-3$ in., $3y-2$ in., and $y+4$ in. respectively. Find its solid contents.

7. By what must 5 be multiplied to make 3 less than n?

8. What is the cost in dollars of 40 electric lights at d dimes each?

9. How many hours will it take to walk m mi. at the rate of 200 ft. a minute?

10. What is the number of days needed by two men to reap a field if p men reap it in q days?

11. B was b years old when C was born. How old will B be when C is $2c$ years old?

12. Show that the sum of the interior angles of a polygon is the same function of n, whether obtained by joining the vertices to a point within the polygon or by drawing all the diagonals from the same vertex. Note in the latter case that the number

of triangles formed is 2 less than the number of sides of the polygon, and in the former case the angles about the point must be subtracted.

13. A man had $\$a$ invested at $x\%$. What would be his total income throughout 3 years?

14. A man invested $\$y$ at 6% and the remainder of his total fortune of $5000 at 5%. What was his total annual income?

15. A man sold a farm that cost him $10,000 at the price of $\$x + 5$ for 50 acres and $\$x$ for the remaining 60 acres, clearing money by the transaction. How much did he clear?

16. A wheelman rode 6 hr. at m mi. an hour. Then he decreased his speed and rode 4 hr. at $m - 3$ mi. an hour. Lastly he rode 2 more hours at $m - 6$ mi. an hour. What was his average speed?

17. In a certain family there are four children, each of which is h years older than the next younger. If y is the number of years in the age of the oldest, what expressions give the age of each of the others, and what is the sum of their ages?

XI. INTEGRAL EQUATIONS

54. An **equation** is a statement that two quantities have the same numerical value (§ 20).

An equation containing an unknown number is essentially an interrogative sentence. The sign "=" is equivalent to the verb "is," or to the phrases, "is the same number as," or "has the same numerical value as."

One of the glories of algebra is that by means of equations one solves with ease problems which students of arithmetic would probably find very puzzling or difficult.

55. The **solution of an equation,** or the finding of that value of its unknown number which makes the two sides of the equation have the same numerical value, is based on the matter already given in this chapter, and on the axioms.

56. An **axiom** is simply a truth assumed as the basis of a study.

57. The Axioms.

1. Addition Axiom: *Equals added to equals give equals.*

2. Subtraction Axiom: *Equals subtracted from equals leave equals.*

3. Multiplication Axiom: *Equals multiplied by the same number, or by equals, give equals.*

4. Division Axiom: *Equals divided by the same number, or by equals* (not zero), *give equals.*

5. Power Axiom: *Equals raised to the same power give equals.*

6. Root Axiom: *Corresponding roots of equals are equal.*

7. General Equational Axiom: *If the same operation be performed on two equal quantities, the results are equal.*

This last axiom holds true in general only for arithmetical numbers.

The axioms will be referred to as "Add. Ax.," "Sub. Ax.," "Mult. Ax.," etc.

8. Fundamental Reasoning Axiom: *Quantities equal to the same quantity are equal to each other.*

9. Axiom of Uniqueness: *The operations of addition, subtraction, multiplication, and division performed on quantities always give unique or "one-valued" results.*

58. Exercise in the Solution of Integral Equations.

1. Find value of x in equation $5x - 2 = 3x + 18$, and verify.

Solution. $5x - 2 = 3x + 18$. (Given equation.)

$5x - 3x = +2 + 18$. (By the subtraction axiom. Thus, the same quantity, $-2 + 3x$, is subtracted from each of the two equals, $5x - 2$ and $3x + 18$, giving the equal remainders, $5x - 3x$ and $2 + 18$.)

Or $2x = 20$. (By adding or simplifying on both sides of the equation.)

$x = 10$. (Division Axiom. The two equals $2x$ and 20 divided by the same number 2 give equals.)

Verification. $5 \times 10 - 2 = 3 \times 10 + 18$, or $48 = 48$.

Studying this solution, we see that when $3x$ disappeared from the right "member" of the equation and appeared in the left "member," its sign was changed from + to −. Similarly, when −2 disappeared from the right member and appeared in the left member, its sign was changed from − to +. Evidently this rule will always hold true, since to "transpose" a quantity we may subtract it from each of the two members of the given equation. This causes it to disappear from the side where it now is, and to appear on the opposite side with its sign changed.

We have the following rule for solving an integral equation:

2. *Transpose all terms containing the unknown number to the left member of the equation and all known terms to the right member, changing the sign of every term transposed. Then add, or "collect," the quantities on each side, as in § 35. To finish the solution, divide both members of the resulting equation by the coefficient of the unknown. Last of all verify the answer found in the given equation.*

Solve and verify in the following, the first four mentally:

3. $2x-3=x+5.$
4. $5x-7=3x+15.$
5. $3p-25=p-9.$
6. $24n-49=19n-14.$
7. $7(m-3)=9(m+1)-38.$
8. $59(r-7)=61(9-r)-2.$
9. $5x+11=16-3x-4x.$
10. $6y-[7y-(8y-18)]=17.$
11. $(x+1)(x+2)=(x-3)(x-4).$

a. One x^2 destroys the other in Ex. 11 when it is transposed. The word "cancel" is often used here, but it is desirable always to use the word "destroy" in such instances, reserving the word "cancel" for canceling the same factor in dividend and divisor or numerator and denominator.

12. $(x+1)(x+1)=[111-(1-x)]x-107.$

59. Problems involving the Solution of Integral Equations. The solution of a problem involves three steps: (1) Letting x or some initial letter represent the unknown number, and constructing the one or more functions described in the problem. (2) Changing a sentence in the problem which says two specified numbers are equal into an equation. (3) Solving the equation found and verifying the answer.

1. One base angle of an isosceles triangle is three times as great as the vertical angle. Find the number of degrees in each.

SOLUTION. Let $x =$ the number of degrees in the vertical angle;
then $3x =$ the number of degrees in each base angle.
Now $x + 3x + 3x = 180.$ (Sentence to equation.)
$7x = 180. \quad \therefore x = 25\frac{5}{7}.$

a. The student may have been puzzled to see where the sentence was that was changed into the equation. The answer is it was implied in the statement of the problem. Strictly speaking, the problem should read: "One base angle of an isosceles triangle is three times as great as the vertical angle. Find the number of degrees in each, knowing that the sum of all three angles is 180°." It was also assumed that the number of degrees in the second base angle is the same as in the first base angle (§ 51, 1). In most problems the sentence is explicitly stated, but often it is implied, as in the preceding one.

2. Two angles are supplementary. One has $d - 1$ degrees and the other has $6(d + 1)$. How many degrees are there in each?

3. How many sides has an equiangular polygon four of whose angles equal seven right angles? See Ex. 2, § 52.

4. The area of the United States in 1850 was 559,452 sq. mi. more than 3 times what it was in 1800, and by 1910, exclusive of Alaska and the islands, it had increased 45,535 sq. mi. over what it was in 1850. Now the total increase from 1800 to 1910 was 2,260,675 sq. mi. What was the area of continental United States at each epoch?

5. The population of continental United States in 1850 was 383,408 less than 6 times what it was in 1790, and in 1910 it was 795,238 less than 4 times what it was in 1850. Now the increase from 1850 to 1910 was 10,992,404 more than 3 times what it was from 1790 to 1850. Find the population at each date and verify.

6. Of the six great continents Australia has 386,000 sq. mi. less area than Europe. Africa has 14,000 sq. mi. less than three times the area of Europe, and North America 2,081,000 sq. mi. less than Africa. South America has 828,000 sq. mi. less than twice the area of Europe, and Asia has 769,000 sq. mi. more than both Americas. The total area of all these continents is 52,153,000 sq. mi. Find the area of each continent.

XII. GRAPHS OF STATISTICAL DATA

60. Statistical Graphs.—If two quantities are related in such way that as one changes in value the other also changes, this elative change can be pictured to the eye by means of a figure r diagram on squared paper.

Squared paper is not ordinary quadrille ruled paper, but speially made, the lines being ruled with great accuracy. The nits of measure employed are commonly those of the metric ystem, the larger squares, ruled with heavier lines, being square entimeters, and the little squares having 2 millimeters on a ide.

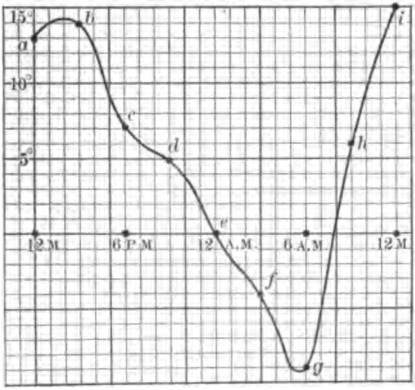
1. To show graphically the changes in temperature throughout he 24 hours of a day, having given the data in the table below.

Solution. On a piece of squared paper we write 12 M. (noon) at any onvenient centimeter intersection. At the right of 12 M. and underneath he sixth line marked with a dot we write 6 P.M.; then 12 A.M. undereath the 12th line, and so on to 12 M. again, underneath the 24th line. assing upward from the first 12 M. point, opposite the fifth horizontal line e write 5°, opposite the tenth line, 10°, etc.

The table shows that at 12 M. the thermometer stood at 13°. To repreent on the diagram this position of the mercury, a dot is placed at *a*, 13 nits above 12 M.; then to represent the position of the mercury as 14° above at 3 P.M., we start at 12 M., move 3 units to the right and then 14 units di-

Table

Hour	Reading	Pt.
12 M.	13°	*a*
3 P.M.	14	*b*
6 P.M.	7	*c*
9 P.M.	5	*d*
12 A.M.	0	*e*
3 A.M.	−4	*f*
6 A.M.	−9	*g*
9 A.M.	6	*h*
12 M.	15	*i*



rectly upward, marking the point reached *b*. To show the position of the mercury at 6 P.M. as 7° above 0, we pass 7 units directly upward from the point marked 6 P.M. to the point *c*; and so on with the remaining data in the table. Through all the points *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, a smooth curve is now drawn. This curve is the *graph* sought.

From the graph the approximate position of the mercury can be read off for any hour of the day. Thus, at 8 A.M. the mercury stood approximately at 1°.

2. Change the following data to tabular form and then construct the graph as in the preceding example: 12 M., 20°; 3 P.M., 15°; 6 P.M., 8°; 9 P.M., 3°; 12 A.M., – 2°; 3 A.M., – 5°; 6 A.M. – 10°; 9 A.M. + 2°; 12 M., 10°.

3. Construct a graph showing the changes of population of continental United States for every ten years from 1790 to 1910 from the following round number data:

1790: 3,900,000
1800: 5,300,000
1810: 7,200,000
1820: 9,600,000
1830: 12,900,000
1840: 17,100,000
1850: 23,200,000
1860: 31,400,000
1870: 38,600,000
1880: 50,200,000
1890: 62,600,000
1900: 76,100,000
1910: 92,000,000

SUGGESTION. Let 1 cm. on the horizontal line represent 10 years, and taking the population for each year to the nearest hundred thousand, let one 2-millimeter unit denote one million on the vertical scale.

4. Find from the graph for Ex. 3 what the population was in 1875. In 1893.

5. The number of immigrants coming into the United States during the fiscal year ending June, 1908 was as follows: July 97,132; August 98,825; September 98,694; October 111,513; November 117,476; December 66,574; January 27,220; February 23,381; March 32,517; April 41,274; May 36,317; June 31,947. Construct graph from these data.

6. Construct other similar graphs, taking data directly from text-books or from one of the statistical almanacs. These almanacs can be had of news or book dealers at small cost, and are very valuable for reference. Among them may be named those issued by the *New York World* and *Chicago Daily News* newspapers.

61. Historical Notes.—Originally everything in algebraical solutions was written out in words. Later, abbreviations for these words were introduced. But it was not till the sixteenth century that symbols came into use to denote quantities, operations, and relations. Even at the present time, in the solution of many problems in arithmetic, words are freely brought into the statement of solutions. In the sixteenth century numerous writers on algebra introduced various signs. Some of these came into general use, while others dropped out. One writer, a Frenchman, Franciscus Vieta (François Viète), more than any other man of his time, made a quite general use of these symbols, and for this reason he has been called the father of algebra. Of course the substance of elementary algebra was then known, but the value of this possession was vastly increased by an appropriate notation.

Vieta * was born near La Rochelle, in France, in 1540, and died in 1603. He practiced law and was connected with both the parliament of Brittany and that of Paris. After 1580 he gave up most of his leisure to the study of mathematics and published numerous books and pamphlets. Being a man of some wealth, he disseminated his writings all over Europe at his own expense. This, no doubt, aided greatly in introducing a uniform symbolism. Vieta used consonants for known numbers and vowels for unknowns, and the vinculum as a symbol of aggregation (these being his personal contributions to our notation), employed + and – for addition and subtraction, but continued to use the letters *aeq.* (*aequalis*) for equals. He used Q (*quadratus*) and C (*cubus*) to denote the second and third powers respectively of the unknown, since exponents had not been thought of at that time.

Thus, for $2x^3 - 3x^2 + 4x = 12$ he would have written

$$2C - 3Q + 4N \text{ aeq. } 12.$$

One day the ambassador from the Low Countries remarked to Henry IV, the king of France, that a countryman of his had propounded a problem to all mathematicians that had not thus far been solved, and asked if a Frenchman could be found to solve it.

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* See Frontispiece.

The king called for Vieta, and though the problem involved the solution of an equation of the 45th degree, Vieta secured two solutions in a few minutes, and gave them to the king written in pencil.

Henry was greatly impressed with the ability of Vieta and thought of him as one who might discover the Spanish cipher then in use by the Spanish government for the transmission of official dispatches. Though this cipher contained over 500 characters, and was changed from time to time, Vieta was soon able to read it, and the French used it greatly to their advantage in the war with Spain then raging. King Philip II complained to the Pope about the matter, charging that the French were using sorcery.

Vieta wrote on algebra, geometry, and trigonometry, but the service he rendered to algebra is his greatest glory.

Leonhard Euler was born in Basel, Switzerland, in 1707, and died in 1783. He studied at the University of Basel, and was a friend of the celebrated Bernouilli family. It was through the younger Bernouillis that he received a call when still a young man by the noted Queen Catherine of Russia to go to St. Petersburg, where he remained many years engaged in teaching and discovery. Later in his life Frederick the Great invited him to go to Berlin, where he continued his life work. Few men in history have been gifted with greater intellectual powers or with a more single devotion to learning.

Euler made a special study of physiology, of medicine, of botany, and of chemistry. He was learned in the classics and could repeat word for word the whole of Vergil's "Æneid," giving the first and last word of each page in the edition of the book he used. He is most noted, however, as a mathematician and astronomer. In mathematics he took many prizes from scientific societies, such as the French Academy.

When but 28 years of age, he solved a problem in three days which the most eminent mathematicians of the time asked for months to solve. The great exertion, however, brought on a fever, which resulted in the loss of the sight of one eye. This did not deter him from work, for only a year or so later he was winning prizes for solving the most difficult problems. In later life, soon

LEONHARD EULER (1707-1783)

after his return to St. Petersburg, a cataract grew over his remaining eye. The operation for its removal was not carefully enough done, and during the last years of his life Euler was nearly blind. It was at this time he was at work on calculations for explaining the inequalities or irregularities in the moon's motions, and these calculations and the data for them he had to carry in his head.

Late in life when almost blind he dictated to a tailor's apprentice, who knew nothing at all of mathematics, his "Introduction to Algebra," a book which is elementary in character, but which has been recognized as a standard text from that day to this. Euler's works if printed in their completeness would fill about 80 huge volumes. His researches aided greatly in the development of the various branches of mathematics, and he stands as one of the greatest mathematicians of all time.

His studies, carried on through a period extending considerably over half a century, much of the time in the face of the greatest obstacles, such as the difficulties of the subject and his own physical defect, set an example of whole-hearted love of knowledge which cannot fail to be an incentive to labor more diligently to all those who come to know of this man and his work.

CHAPTER II

THEOREMS IN MULTIPLICATION AND DIVISION, AND FACTORING

I. THEOREMS

62. A **theorem in algebra** is a fundamental truth expressed in *words*, whereas a **formula** is such a truth expressed in symbols. (See § 52.)

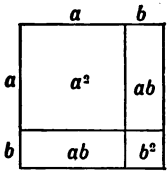
a. In the proofs that follow, and hereafter, we will use heavy-faced type to denote *any* algebraic quantities. Thus, $\mathbf{a}$ may stand for $2\,ab$, or for $c+d$, and $\mathbf{b}$ may stand for $3\,c^2$, or $2\,m+3\,n-5\,p$, etc. The letters in ordinary italic type are understood to stand each for some number. Letters in heavy-faced type are to be understood as capable of denoting either a single letter or any combination of letters, figures, and signs, denoting a quantity. To be able to generalize in this way is one of the strong features of algebra.

63. THEOREM I. *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.*

PROOF. Let $\mathbf{a}$ and $\mathbf{b}$ be any two quantities. Then by actual multiplication:

$$\begin{array}{lll} \mathbf{a} & +\mathbf{b} & \\ \mathbf{a} & +\mathbf{b} & \\ \hline \mathbf{a}^2 & +\mathbf{ab} & \\ & +\mathbf{ab} & +\mathbf{b}^2 \\ \hline \mathbf{a}^2 & +2\,\mathbf{ab} & +\mathbf{b}^2 \end{array}$$

Or, $(\mathbf{a}+\mathbf{b})^2=\mathbf{a}^2+2\,\mathbf{ab}+\mathbf{b}^2$.



Beside the multiplication we see a geometrical proof of this theorem. The square constructed on the sum of the lines $\mathbf{a}$ and $\mathbf{b}$ equals the square $\mathbf{a}^2$ constructed on $\mathbf{a}$, plus two rectangles $\mathbf{a}$ long and $\mathbf{b}$ wide, plus the square $\mathbf{b}^2$ constructed on $\mathbf{b}$.

The student should practice *pointing* to the different characters in the formula $(a+b)^2 = a^2 + 2ab + b^2$ which correspond to the words of the theorem as he repeats the theorem. *This course should be followed in the study of all the theorems.* Notice that in reading algebraical expressions very often we do not go regularly from left to right as in ordinary print.

Expand mentally the following by using the theorem:

1. $(2a+3b)^2$.

Solution. $(2a+3b)^2 = (2a)^2 + 2(2a \times 3b) + (3b)^2$.
$= 4a^2 + 12ab + 9b^2$. *Ans.*

2. $(5a+2b)^2$. **3.** $(7a+5b)^2$.

4. $(3a^2+2y)^2$. **5.** $(3x+4y)(3x+4y)$.

6. $(2m^3n+1)^2$. **7.** $(a^3+2b^2)(a^3+2b^2)$.

8. $(\frac{1}{2}x+\frac{3}{5}y)^2$. **9.** $(7+\frac{1}{2})^2$.

10. $(15+\frac{1}{8})(15+\frac{1}{8})$. **11.** $(100+2)^2$.

12. $(25+3)^2$. **13.** $(18+\frac{2}{3})^2$. **14.** $(7\frac{1}{8})^2$.

15. $(16\frac{1}{4})^2$. **16.** $(12\frac{5}{6})^2$. **17.** $(9\frac{2}{5})^2$.

18. 24.5^2. **19.** 8.2^2. **20.** 25.5^2.

21. 100.5^2. **22.** 80.4^2. **23.** 20.8^2.

a. The object in giving so many arithmetical problems here is to accustom the pupil to apply his algebra to arithmetic. Such calculations can often be performed mentally and much more quickly than by ordinary multiplication; or, they can be used to check the accuracy of ordinary multiplication.

Factor the following quantities (this being the reverse operation to that we have been performing):

24. $9x^2+30xy+25y^2$.

Solution. $\sqrt{9x^2} = 3x$; $\sqrt{25y^2} = 5y$. *Possible Ans.* $(3x+5y)^2$.

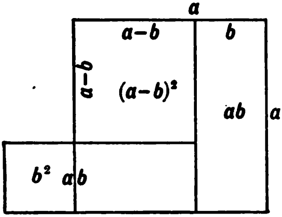
Check. $(3x+5y)^2 = (3x)^2 + 2(3x \times 5y) + (5y)^2 = 9x^2 + 30xy + 25y^2$.

25. $4m^2+12mn+9n^2$. **26.** $x^2+10x+25$.

27. $a^2b^2+2abcd+c^2d^2$. **28.** $36x^4+84x^2y^2+49y^4$.

29. $x^2+x+\frac{1}{4}$. **30.** $a^{2n}+4a^n+4$.

64. **Theorem II.** *The square of the difference of two quantiies equals the square of the first, minus twice the product of the first nd second, plus the square of the secnd.*



Proof. Let a and b represent any two uantities. By actual multiplication, we ave $(a-b)^2 = a^2 - 2ab + b^2$.

Proof by Geometry. $a^2 + b^2$ (or the rea of the large square + the area of the mall square below and to the left of the large quare) diminished by the two ab rectangles equals the square on $a - b$.

Expand the following quantities by using Theorem II:

1. $(2m^2 - 3n)^2$.

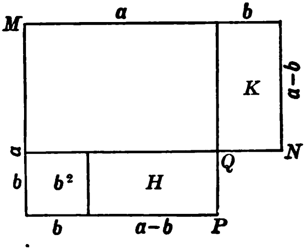
Solution. $(2m^2 - 3n)^2 = 4m^4 - 12m^2n + 9n^2$. *Ans.*

2. $(3mn - 5)^2$. 3. $(3abc - bcd)^2$. 4. $(100 - 3)^2$.
5. $(1 - \frac{2}{3}x^2)^2$. 6. $(16 - \frac{1}{4})^2$. 7. $(.4x^5 - .2)^2$.
8. $(12 - \frac{1}{8})^2$. 9. $(112 - \frac{1}{3})^2$. 10. $(87\frac{2}{3})^2$.
11. $(24\frac{4}{5})^2$. 12. $(17\frac{5}{8})^2$. 13. $(49\frac{5}{6})^2$.

Factor (reverse operation) the following:

14. $a^2 - 8a + 16$. (See § 63, Ex. 24.)
15. $m^4 - 20m^2 + 100$. 16. $4x^2y^2 - 20xyz + 25z^2$.
17. $36x^4 - 60x^2y + 25y^2$. 18. $1 - 6b^2c^3 + 9b^4c^6$.

65. **Theorem III.** *The product of the sum and the difference of two quantities equals the difference of their squares; and, conversely, the difference of the squares of two quantities equals the product of their sum and their difference.*



Proof. Let a and b represent any two quantities.

By actual multiplication

$$(a+b)(a-b) = a^2 - b^2.$$

Proof by Geometry.

$$H = K = (a-b)b.$$

Then, $MN = MQ + H = MP - b^2$.

or, $(a+b)(a-b) = a^2 - b^2$.

Expand the following quantities by using Theorem III:

1. $(2a + 3b)(2a - 3b)$.

Solution. $(2a + 3b)(2a - 3b) = (2a)^2 - (3b)^2 = 4a^2 - 9b^2$. *Ans.*

2. $(3 + x)(3 - x)$.
3. $(m^6 - n^6)(m^6 + n^6)$.
4. $(3a^2 + 7b^3)(3a^2 - 7b^3)$.
5. $(a^m + b^n)(a^m - b^n)$.
6. 58×62, or $(60 - 2)(60 + 2)$.
7. $(17 + \frac{1}{3})(17 - \frac{1}{3})$.
8. 73×67.
9. $(26\frac{3}{4} \times 33\frac{1}{4})$.
10. $a^2 = (a + b)(a - b) + b^2$. (By transposing b^2.)

Thus, $67^2 = 70 \times 64 + 9 = 4489$.

11. Square in same way 27; 54; 89; 117.

Factor (reverse operation) the following:

12. $9a^2 - 16b^2$.
13. $25b^4 - 36$.
14. $4m^6 - 1$.
15. $49a^4 - \frac{1}{9}b^2$.
16. $a^{2m} - b^{2n}$.
17. $\frac{1}{4}x^2y^2z^2 - 25n^2$.

66. Theorem IV. *The product of two binomials having a common term equals the square of the common term, and the algebraic sum of the other two terms times the common term, and the algebraic product of the other two terms.*

Proof. Let x, a, and b represent any three quantities. Then

$$\begin{array}{l} x + a \\ x + b \\ \hline x^2 + ax \\ \quad + bx + a \\ \hline x^2 + (a + b)x + ab \end{array} \qquad \begin{array}{l} x - a \\ x - b \\ \hline x^2 - ax \\ \quad - bx + ab \\ \hline x^2 + (-a - b)x + ab \end{array} \qquad \begin{array}{l} x - a \\ x + b \\ \hline x^2 - ax \\ \quad + bx - ab \\ \hline x^2 + (-a + b)x - ab \end{array}$$

Expand the following quantities mentally by using Theorem IV:

1. $(x + 5)(x - 6)$.

Solution. $(x + 5)(x - 6) = x^2 + (5 - 6)x - 30 = x^2 - x - 30$. *Ans.*

2. $(x + 2)(x + 3)$.
3. $(x - 4y)(x - 6y)$.
4. $(ax + 6)(ax - 9)$.
5. $(x^2 - 3xy)(x^2 + xy)$.
6. $(2a + 3)(2a + 7)$.
7. $(3ab - 7)(3ab - 5)$.
8. $(2m^2 + 5n)(2m^2 - 10n)$.
9. $(5c^n - 8d)(5c^n + 9d)$.
10. $(90 - 4)(90 + 3)$.
11. $(70 + 4)(70 + 5)$.

12. 64×67. **13.** 52×58. **14.** 47×54. **15.** 29×33.
16. 18×23. **17.** 97×105. **18.** 997×1004. **19.** 88×94.
20. 37×47. **21.** 16×29. **22.** 87×95. **23.** 63×69.

Factor (reverse operation) the following:

24. $x^2 + 11x - 26$.

Solution. We find two numbers 13 and -2 whose *product* is -26 and *sum* is 11. $(x+13)(x-2)$. *Ans.*

25. $z^2 + 13yz + 12y^2$. **26.** $b^2 - 30b + 200$.
27. $a^{2n} - 18a^n + 45$. **28.** $4x^{2a} + 4x^a - 15$.
29. $y^8z^4 - 5y^4z^2n - 84n^2$. **30.** $y^2 + \frac{1}{2}y - \frac{3}{16}$.

67. Theorem V. *The cube of the difference of two quantities equals the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.*

(In the cube of the *sum* of two quantities all the terms are positive.)

Proof. By actual multiplication. Left to pupil.

1. $(2a^2 - 5b^3)^3$.

Solution. $(2a^2 - 5b^3)^3 = (2a^2)^3 - 3(2a^2)^2(5b^3) + 3(2a^2)(5b^3)^2 - (5b^3)^3$
$= 8a^6 - 60a^4b^3 + 150a^2b^6 - 125b^9$. (§ 21, 1.)

2. $(2a - 4b^2)^3$. **3.** $(2a^2 - \frac{1}{2}b)^3$. **4.** $(a^n - \frac{1}{4}b^2)^3$.

Factor (reverse operation) the following:

5. $27a^3 - 108a^2b + 144ab^2 - 64b^3$.

Solution. $\sqrt[3]{27a^3} = 3a$; $\sqrt[3]{-64b^3} = -4b$. *Possible Ans.* $(3a - 4b)^3$.

Check. $(3a - 4b)^3 = (3a)^3 - 3(3a)^2(4b) + 3(3a)(4b)^2 - (4b)^3$.
$= 27a^3 - 108a^2b + 144ab^2 - 64b^3$.

6. $125x^3 - 75x^2 + 15x - 1$.

7. $8a^6 + 72a^4b^2 + 216a^2b^4 + 216b^6$.

68. Theorem VI. *The square of the sum of any number of quantities equals the sum of their squares and the sum of twice the product of each quantity by each quantity that follows it.*

Proof. We will limit the proof to trinomials and quadrinomials, but the theorem holds generally. By actual multiplication

$$(a+b+c)^2 = a^2+b^2+c^2+2\,ab+2\,ac+2\,bc.$$

$$\begin{aligned}(a-b-c+d)^2 &= [a+(-b)+(-c)+d]^2\\ &= a^2+b^2+c^2+d^2-2\,ab-2\,ac+2\,ad+2\,bc-2\,bd-2\,cd.\end{aligned}$$

Expand the following by Theorem VI:

1. $(a-2\,b+5\,c)^2$.

Solution. $(a-2\,b+5\,c)^2 = a^2+4\,b^2+25\,c^2-4\,ab+10\,ac-20\,bc.$

2. $(3\,a-4\,m+2\,n)^2$.

3. $(2\,a^2-3\,bc+7\,b^2)^2$.

4. $(x-2\,y+z-5\,n)^2$.

5. $(a^m-2\,b^m-3\,c^m-2\,d^m)^2$.

69. Theorem VII. Divisibility Theorem.* *If a polynomial in x* (§ 15) *equals* 0 *when $x=a$, then the polynomial is exactly divisible by $x-a$.*

1. Show by the divisibility theorem that $x^2-7\,x+10$ is exactly divisible by $x-2$.

Solution. Putting $x=2$, $x^2-7\,x+10=4-14+10=0$.

Check. $(x^2-7\,x+10)\div(x-2)=x-5$.

2. Show that $x^2-27\,x-124$ is exactly divisible by $x+4$, and check.

Suggestion. Put $x=-4$. If polynomial equals 0 when this value of x is substituted, then by the theorem it is exactly divisible by $x-(-4)$, or by $x+4$.

3. By what is $2\,x^3+3\,x^2-4\,x-1$ exactly divisible?

Suggestion. Noting sum of coefficients, will $x=1$ answer?

---

[*] **Proof.** Let p = any polynomial in x, q = the quotient when p is divided by $x-a$, and r = the remainder, the division being continued until the remainder does not contain x.

Then, $p=q(x-a)+r$ (Since dividend = quotient × divisor + remainder.)

Now if $x=a$, $p=q(a-a)+r$. (By substituting a for x.)

Or, $0=q\times 0+r$. (Since $p=0$, by hypothesis, when $x=a$).

Hence, $r=0$,

or there is *no* remainder when p is divided by $x-a$.

4. By what is $x^3+6x^2+3x-10$ exactly divisible on inspection?

5. By what is $x^4+6x^3+3x^2+x+3$ exactly divisible? Put $x=+1$, -1, $+3$, and -3 in turn, these being the factors of the last term.

70. THEOREM VIII.* *The difference of any same powers of two quantities is exactly divisible by the difference of the quantities.*

For example, x^2-y^2, x^3-y^3, x^4-y^4, etc. are each exactly divisible by $x-y$.

71. THEOREM IX.† *The difference of the same even powers of two quantities is exactly divisible by both the difference and the sum of the quantities.*

Thus, x^2-y^2, x^4-y^4, x^6-y^6, etc. are each exactly divisible by both $x-y$ and $x+y$.

72. THEOREM X.‡ *The sum of the same odd powers of two quantities is exactly divisible by the sum of the quantities.*

Thus, x^3+y^3, x^5+y^5, x^7+y^7, etc. are exactly divisible by $x+y$.

---

* PROOF. Let $x^n - a^n$ represent the difference of any same integral powers of two quantities, x and a. Now, if $x=a$,

$$x^n - a^n = a^n - a^n = 0,$$

and therefore, by § 69, $x^n - a^n$ is exactly divisible by $x-a$, no matter what the values of x and a are.

† PROOF. The difference of the same even powers is exactly divisible by the difference of the quantities by Theorem VIII. It remains to prove that the difference of the same even powers is also divisible by the sum of the quantities.

Let $x^n - a^n$ represent the difference of the same even powers of two quantities. Then, if $x=-a$, $\quad x^n - a^n = (-a)^n - a^n = a^n - a^n = 0,$

because n is even; and hence, by the divisibility theorem, § 69, $x^n - a^n$ is exactly divisible by $x-(-a)$, or $x+a$.

‡ PROOF. Let $x^n + a^n$ represent the sum of the same odd powers of two quantities x and a. Then, if $x=-a$,

$$x^n + a^n = (-a)^n + a^n = -a^n + a^n = 0,$$

because n is odd; and hence by the divisibility theorem, § 69, $x^n + a^n$ is exactly divisible by $x-(-a)$, or $x+a$.

73. The Factoring of Binomials by Theorems VIII–X.* *Theorems VIII–X give* **one** *factor of any binomial which is the sum or difference of the same powers of two quantities. The other factor can be found by long division. Or, to find the remaining factor, divide the first term of the given binomial by the first term of the known factor, for the first term of the required factor. To find the next term of the required factor, divide its first term just found by the first term of the known factor and multiply the quotient by the second term of the known factor, ignoring signs. The third term is found from the second in the same manner as the second was found from the first; and so on to the last term.*

Whenever the known factor is a **difference**, *all the signs of the other factor are positive; when it is a* **sum**, *the signs of the second factor will be alternately positive and negative.*

The reasons for this rule can be easily seen by referring to actual long divisions.

Separate the following into their prime factors, proving the answers by actual multiplications if there is any uncertainty as to their correctness. Some or all should be performed mentally.

1. $m^3 - n^3$.
2. $8x^3 - y^3$.
3. $9x^2 - 16y^2$.
4. $x^5 - y^5$.
5. $27m^3 + 64n^3$.
6. $x^6 - y^6$.
7. $a^7 + b^7$.
8. $16c^4 - 81d^4$.
9. $a^6 + 125b^6$.
10. $a^9 + 8b^9$.
11. $27a^{12} + 8b^9$.
12. $a^{15} - \frac{1}{8}b^{18}$.

a. Theorems VIII–X cover the ground of the divisibility of binomials by *binomials.* Expressions which do not fall under one or other of these theorems are not divisible by binomial divisors. For example, $a^3 - b^3$, $a^5 - b^5$, etc. are not exactly divisible by $a + b$. Also $a^2 + b^2$, $a^4 + b^4$, $a^6 + b^6$, etc. are not exactly divisible by either $a + b$ or $a - b$. Test the truth of these statements by actual divisions, *and then state the results in the form of theorems.* Every student of algebra should know these truths.

b. It must be carefully noted, however, that it often happens that quantities which cannot be factored when tested in one way can be factored when tested in another.

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[*] The theorems just enunciated include the cases of the factoring of the difference of two cubes and the sum of two cubes. Thus,

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2). \qquad \text{(By Theorem VIII, and § 73.)}$$

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2). \qquad \text{(By Theorem X, and § 73.)}$$

For example, while $a^6 + b^6$ is not divisible by either $a + b$ or $a - b$, it is visible by $a^2 + b^2$ by Theorem X, since it can be written $(a^2)^3 + (b^2)^3$. gain, while such an expression as $a^4 + 4b^4$ is not divisible by any real *nomial* divisor, it will be shown later that it is divisible by a *trinomial* visor.

74. Promiscuous Exercise in the Use of the Theorems. Solve entally.

Expand the first 23, and factor the remaining exercises. Quote e theorem used in each exercise.

1. $(2a + b)^2$.
2. $(a - 3b)^2$.
3. $(a + 2b)(a - 2b)$.
4. $(x - y)^3$.
5. $(x + 5)(x - 4)$.
6. $(a - b)(a - b)$.
7. $(3a^2 - 2b^2)^2$.
8. $(x - 7y)(x - 11y)$.
9. $(m^2 - 7n)(m^2 + 12n)$.
10. $(3a - 1)^3$.

1. $(2a^2 - 5b^2)(2a^2 + 5b^2)$.

12. $(2a - 11)(2a - 10)$.

3. $(3a^m - 2b^n)^2$.

14. $(2a^m + 7)(2a^m - 7)$.

5. $(x^3 - y^3)(x^3 + 21y^3)$.

16. $(2m - 3)^3$.

7. $(3x - 8)(3x + 6)$.

18. $(\frac{1}{2}m - \frac{1}{3}n)(\frac{1}{2}m + \frac{1}{3}n)$.

9. $(100 - 2)^2$.

20. 77×83.

21. $(2x - y - 3z)^2$.

22. $(3m^2 - 2n^3 + p^2)^2$.

23. $(a - 2b - 3c + 4)^2$.

24. $64x^3 - y^3$.

25. $9a^2 + 30ay + 25y^2$.

26. $a^5 - 32b^5$.

27. $x^2y^2 - 23xy + 60$.

28. $x^4 - 25y^4$.

29. $x^3 - 9x^2 + 27x - 27$.

30. $1 - (x - y)^2$.

31. $16a^2b^2c^2 + 24ab^2c^3 + 9b^2c^4$.

32. $a^3b^3 + 512$.

33. $a^2b^2 - 13abc + 40c^2$.

34. $a^9 - x^6$.

35. $a^4y^2 + 7a^2yz - 8z^2$.

36. $a^{2n} - b^{2n}$.

37. $m^{4p} - 3m^{2p} - 70$.

38. $8a^3 - 27b^3$.

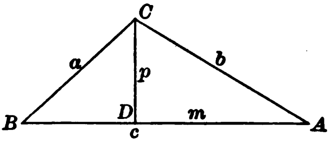
39. $8x^3 - 60x^2y + 150xy^2 - 125y^3$.

40. $1 - (x - y)^3$.

41. $(a + b)^2 - 9c^2$.

75. Practical Applications of the Theorems of Multiplication.

1. *Prove that the square of the side opposite an acute angle in a triangle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other on that side.*



To prove $a^2 = b^2 + c^2 - 2\,cm$.

PROOF. $a^2 = p^2 + \overline{BD}^2$ (§ 51.)

$= p^2 + (c - m)^2$

$= p^2 + c^2 - 2\,cm + m^2$. (By Theorem II.)

$\therefore a^2 = b^2 + c^2 - 2\,cm$. Q.E.D. (Because $p^2 + m^2 = b^2$ by Pyth. Prop.)

2. Find the square of the side opposite an obtuse angle in a triangle.

3. Pythagoras stated that if $2\,a + 1$ and $2\,a(a + 1)$ are the legs of a right triangle, $2\,a^2 + 2\,a + 1$ is the hypotenuse. Prove that this is true by showing that the sum of the squares of the first two equals the square of the last.

4. Plato gave $4\,a$, $4\,a^2 - 1$, and $4\,a^2 + 1$ for the sides and hypotenuse, and Brahmagupta gave $2\,ab$, $a^2 - b^2$, and $a^2 + b^2$; verify their statements.

5. Put $a = 1, 2, \ldots$ in Ex. 3, 4, getting sets of integral numbers whose sides form right triangles.

6. Find how far a body will fall in the last second of its fall by calculating (simplifying) $\frac{1}{2}\,gt^2 - \frac{1}{2}\,g(t-1)^2$.

II. FACTORING

76. Factoring is separating a quantity into other quantities whose product equals the given quantity. The quantities already factored and those to be factored in this chapter are rational integral quantities, or quantities that do not contain root expressions which cannot be found exactly.

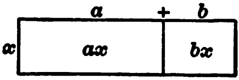
Factoring is an important topic. The difficulty of mastering the subject is greater than learning the fundamental operations because of the variety of processes and kinds of problems.

In the study that follows we will classify quantities into monomials, binomials, trinomials, quadrinomials, and multinomials. We will take up first a case which stands by itself, viz., that of monomial factors contained in polynomials.

77. Monomial Factors in Polynomials. A monomial is a factor of a polynomial if it is contained *in every term* of the polynomial.

Type forms: $ax + bx = (a + b)x$;

$3mx + 3my - 3mz = 3m(x + y - z)$.



Geometrical Meaning of Factoring. In the factored form $(a+b)x$, the area is expressed as the product of length by breadth; in the unfactored form, $ax + bx$, the area of each part is found and these parts are added. See p. 18.

1. Factor $6a^3 - 2am^2$ and prove answer by multiplication.

Division Solution. $2a\overline{)6a^3 - 2am^2}$
$\quad 3a^2 - m^2$

Proof.
$$\begin{array}{r} 3a^2 - m^2 \\ 2a \\ \hline 6a^3 - 2am^2. \end{array}$$

Direct Solution. $6a^3 - 2am^2 = 2a(3a^2 - m^2)$.

Factor the following, using both the division form and the direct solution form:

2. $4a^3 - 2ab$.
3. $20x^2y - 5y$.
4. $2m^3 - 6m^2n + 10mn$.
5. $a^2 - ab$.
6. $8a^3 - 12a^2 - 4$.
7. $ab - ac + ad - abc$.
8. $a(m + n) - 5b(m + n)$

Solution. $m+n\underline{|\ a(m+n) - 5b(m+n)}$
$\qquad a \qquad\quad -5b$

9. $2b(3x - y) + 3c(3x - y) + 2d(3x - y)$.

a. Throughout factoring the first step to be taken is to remove the monomial factors from the given quantities, if any are present. The other factor can then be separated into its factors. The presence of the monomial factor disguises many problems, and until it is removed, the student will not know how to proceed. This fact should not be forgotten.

1. Monomials

78. Factoring of Monomials. A monomial such as $3a^2b^3c$ is regarded as already factored. A quantity such as $48m^3n$ is factored when its coefficient is factored. Thus, $48m^3n = 2^4 \times 3m^3n$.

2. Binomials

79. Factoring of Binomials. Two different classes of problems may be distinguished, of which the first is altogether the more important.

I. *Binomials factored directly by one of the theorems: converse in Theorem III, and Theorems VIII, IX, X.*

See, for examples, §§ 65, 73.

II. *Binomials factored by inserting a quantity between the terms of the given binomial, and then subtracting the same quantity in a fourth term, thus changing the given binomial into a quadrinomial which is the difference of two squares.* See Ex. 13 below.

Factor the following by one of the theorems mentioned above:

1. $16x^4 - 9y^2$.
2. $x^3 - 27y^3$.
3. $a^4 - b^4$.
4. $8m^6 + 125n^6$.
5. $a^3b^3c^3 - 1$.
6. $z^5 + 1$.
7. $x^9 + y^6$.
8. $a^{12} - 16b^8$.
9. $16x^4 - 625$.
10. $4x^6 + 108z^6$.
11. $96m^4 - 486$.
12. $x^7 + 64xz^6$.
13. Factor $a^4 + 4b^4$ and prove by multiplication.

Solution

(1) $(a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2$.	(The term $4a^2b^2$, which is added and then subtracted, is twice the product of the square roots of a^4 and $4b^4$. Notice $4a^2b^2$ is chosen so that $a^4 + 4a^2b^2 + 4b^4$ is the square of a binomial as obtained by Theorem I, § 63.)
(2) $(a^2 + 2b^2)^2 - (2ab)^2$.	(By writing each part in the form of a square.)
(3) $(\overline{a^2 + 2b^2} + 2ab)(\overline{a^2 + 2b^2} - 2ab)$.	(Factoring by converse of Theorem III, § 65.)
(4) $(a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$.	(Removing vincula and rearranging.)
(5) Prove by actual multiplication.	

14. $a^4 + 64b^4$.
15. $x^4 + 324y^4$.
16. $4m^4 + 81p^4$.

3. Trinomials

80. The Factoring of Trinomials. Three different classes of problems may be distinguished, of which the first is much the most important.

I. *Trinomials factored by the trial method.* See Exs. 3, 4.

a. In certain special cases the trial method is materially aided by making use of Theorems I, II, IV. For instance, to factor a trinomial square, extract the square root of the first and last terms, and square the sum or difference of the roots by Theorems I and II to see if the result is the given trinomial. To factor a trinomial, the product of any two binomials having a common term, search for two quantities whose product is the third term of the given (arranged) trinomial, and whose algebraic sum times the square root of its square term equals the middle term of the given trinomial.

II. *Trinomial factored by changing the given trinomial into a quadrinomial which is the difference of two squares.* See Ex. 16.

III. *Trinomials factored by the divisibility theorem.* See Ex. 20.

1. Factor $9x^4 + 30x^2 + 25$, using Theorem I. See § 63, Ex. 24.

2. Factor $y^2 - 4yz - 45z^2$, using Theorem IV. See § 66, Ex. 24.

3. Factor $15a^2 - 77am + 10m^2$ by the trial method.

SOLUTION.

$$\begin{array}{rrr} 3a & -5m & \\ 5a & -2m & \\ \hline 15a^2 & -25am & \\ & -6am & +10m^2 \\ \hline 15a^2 & -31am & +10m^2 \end{array} \qquad \begin{array}{rrr} 3a & -2m & \\ 5a & -5m & \\ \hline 15a^2 & -10am & \\ & -15am & +10m^2 \\ \hline 15a^2 & -25am & +10m^2 \end{array} \qquad \begin{array}{rrr} 15a & -2m & \\ a & -5m & \\ \hline 15a^2 & -2am & \\ & -75am & +10m^2 \\ \hline 15a^2 & -77am & +10m^2 \end{array}$$

Evidently this method consists in selecting coefficients for a pair of factors in such a way that the product of these factors will have the same coefficients for its first and last terms as those of the given trinomial. Then, by actual multiplication, the product is found and compared with the given trinomial. If the middle coefficients differ, another set of trial coefficients must be selected, and so on, till a set of coefficients and signs is found which gives the desired product.

b. If the given trinomial contains no monomial factor, neither can either of its factors. For, if one of two factors contains a factor, their product will contain this factor.

Thus, in the problem just solved it was useless to try the second set o factors $3a - 2m$ and $5a - 5m$, since $5a - 5m$ contains the factor 5 and th given trinomial does not.

We will give next another kind of trial method for factorin trinomials whose coefficients are large or give rise to many combinations of factors.

4. Factor $12x^2 - 41x - 15$.

Solution. $(12x)^2 - 41(12x) - 12 \times 15$. (Given quantity multiplied b the coefficient of x^2.)

This expression is now factored by the method of § 66. We seek for tw numbers whose *product* is -180, or the product of the first and las coefficients of the given quantity, and whose *sum* is -41, or the coefficient o x in the given quantity. Starting with the divisor 2 and trying each diviso of -180 and its quotient, we look for a pair of factors whose sum is -41.

In this way we get 4 and -45. Then we write

$$(12x + 4)(12x - 45).$$

But we introduced the factor 12 above, and now must remove it from th factors found. Taking 4 out of the first factor and 3 out of the second, w have $(3x + 1)(4x - 15)$. *Ans.*

Proof. $(3x + 1)(4x - 15) = 12x^2 - 41x - 15$.

5. $x^4 + x^2 - 30$. **6.** $y^2z^2 + 4yz + 4$.

7. $5a^4 - 10a^2b + 5b^2$. **8.** $24x^2 - 29xy - 4y^2$.

9. $2c^2 - 13cd + 6d^2$. **10.** $16a^{2n} - 24a^n + 9$.

11. $\frac{9}{16}y^2 - \frac{15}{2}y + 25$. **12.** $x^{2m} - 19x^m + 90$.

13. $20 - 9x - 20x^2$. **14.** $24a^2b^2c^2 - 37abc - 72$.

15. $25m^4 - 41m^2n^2 + 16n^4$. Factor each of the factors.

16. Factor $9x^4 + 38x^2y^2 + 49y^4$ by changing to a quadrinomia which is the difference of two squares.

Solution

(1) $(9x^4 + 42x^2y^2 + 49y^4) - 4x^2y^2$. (The term $42x^2y^2$ is found by taking twice the product of th square root of $9x^4$ and of $49y$ Then $4x^2y^2$ has to be subtracte in a fourth term to leave th given quantity unaltered.)

(2) $(3x^2+7y^2)^2-(2xy)^2$. (Writing each part in form of square.)

(3) $[(3x^2+7y^2)+2xy][(3x^2+7y^2)-2xy]$. (Factoring by converse of Theorem III, § 65.)

(4) $[3x^2+2xy+7y^2][3x^2-2xy+7y^2]$. (Removing parentheses and rearranging.)

(5) Prove by actual multiplication.

17. Factor $9a^4+21a^2c^2+25c^4$ and prove.

18. Factor $a^4+a^2b^2+b^4$ and prove.

19. Factor $16a^4-17a^2b^2+b^4$ and prove.

20. Factor $x^3-19x-30$ by the divisibility theorem.

Solution. — We try the factors of 30.

If $x=\pm 1$, $x^3-19x-30=-48$, or -12.

If $x=\pm 2$, $x^3-19x-30=-60$, or 0; thus $x-(-2)$ is a factor.

If $x=\pm 3$, $x^3-19x-30=-60$, or 0; thus $x-(-3)$ is a factor.

If $x=\pm 5$, $x^3-19x-30=0$, or -60; thus $x-5$ is a factor.

Hence, $x^3-19x-30=(x+2)(x+3)(x-5)$. Prove by actual multiplication.

21. x^3-7x+6. **22.** $y^3-13y+12$. **23.** $t^3-21t-20$.

4. Quadrinomials

81. The Factoring of Quadrinomials. Four different classes of problems may be distinguished.

I. *Quadrinomials the product of two binomials, solved by inserting one pair of terms in one parenthesis, and the remaining pair in another parenthesis, and then, after removing monomial factors from each, dividing by a binomial factor now apparent.* See Ex. 1.

II. *Quadrinomials the difference of two squares, solved by placing three terms in parenthesis, and then factoring by converse of Theorem III.* See Exs. 8, 13.

III. *Quadrinomials the cube of a binomial, solved by extracting the cube root of the first and last terms, and checking the suggested answer by cubing the sum or difference of the roots by Theorem V.* See Ex. 18.

IV. *Quadrinomials factored by the divisibility theorem.* See Ex. 24.

1. Factor $3a^2x^3 - 4acx^2 - 6abx + 8bc$.

Solution. $(3a^2x^3 - 4acx^2) - (6abx - 8bc)$ (§ 37).

$$ax^2(3ax - 4c) - 2b(3ax - 4c)$$

$$\begin{array}{l|l} 3ax - 4c & ax^2(3ax - 4c) - 2b(3ax - 4c) \\ \hline & ax^2 \qquad\qquad -2b. \end{array} \qquad (\S\ 77,\ 8)$$

Proof. By multiplication.

Note. When the first term to be put in the second parenthesis is negative, the rule is to make the sign before the parenthesis negative. If one arrangement of the terms in parentheses fails to show a common binomial factor, try another. When the terms of the binomial in the second parenthesis have signs opposite to those in the first, change the sign before the second parenthesis, changing the signs inside at the same time.

Factor the following:

2. $ac - bd + bc - ad$.
3. $a^3 + a^2 + 2a + 2$.
4. $xy - 3y + 2x - 6$.
5. $cdx^2 - cxy + dxy - y^2$.
6. $15a^3 + 12a^2 - 10a - 8$.
7. $abcxy - b^2dy - acdx^2 + bd^2x$.
8. Factor $a^2 + 2ab - x^2 + b^2$, and prove.

Solution

(1) $(a^2 + 2ab + b^2) - x^2$. (Placing the three terms which form the square of a binomial in parenthesis.)

(2) $(a + b)^2 - x^2$. (Factoring the trinomial.)

(3) $(\overline{a + b} + x)(\overline{a + b} - x)$. (By converse of Theorem III.)

(4) $(a + b + x)(a + b - x)$. (Removing vincula.)

(5) **Proof.** By multiplication.

Factor the following:

9. $b^2 - 1 - 2ab + a^2$.
10. $4a^2 - 9b^2 + c^2 + 4ac$.
11. $x^2 - z^2 - 2xy + y^2$.
12. $(a - b)^2 - 4c^2$.
13. $4bc + a^2 - 4b^2 - c^2$.

Solution

(1) $a^2 - (4b^2 - 4bc + c^2)$. (The monomial square is here positiv and must come first; the trinomia square must be inserted in a parenthesi preceded by minus sign.)

(2) $a^2-(2b-c)^2$. (Factoring the trinomial.)

(3) $(a+\overline{2b-c})(a-\overline{2b-c})$. (By converse of Theorem III.)

(4) $(a+2b-c)(a-2b+c)$. (Removing vincula.)

(5) Prove by actual multiplication.

14. $4a^2+6bc-9b^2-c^2$.

15. $12bc+4b^4-9b^2-4c^2$.

16. $24ab-4a^2+9x^2-36b^2$.

17. $2ab-a^2-b^2+1$.

18. $27c^2-12a^2+36ab-27b^2$.

19. $3x^2y-12xy^2+12y^3-3x^6y$.

20. $64x^6-144x^4y^2+108x^2y^4-27y^6$. (See § 67, Ex. 5.)

21. $8x^3-60x^2y+150xy^2-125y^3$.

22. $216a^6b^3+108a^4b^2c+18a^2bc^2+c^3$.

23. $x^3+5x^2-18x+8$. (See § 69.)

24. $12x^3-40x^2+13x+30$.

Solution. A factor of the given quantity may be of the form $ax-b$, of which a is exactly contained in 12, and b is exactly contained in 30.

If $ax-b=0$, $x-\frac{b}{a}=0$ (Div. Ax.). If, when $\frac{b}{a}$ is substituted for x, the given polynomial in x reduces to 0, $\frac{b}{a}$ is a root. Thus, to find a factor of $12x^3-40x^2+13x+30$, we will substitute for x fractions whose numerators are factors of 30, and whose denominators are factors of 12, as $\frac{6}{6}$, $\frac{6}{3}$, $\frac{6}{4}$, etc.

If $x=\pm 1$, $12x^3-40x^2+13x+30=15$, or -35.

If $x=\pm 2$, $12x^3-40x^2+13x+30=-8$, or -252.

If $x=\pm\frac{3}{2}$, $12x^3-40x^2+13x+30=0$, or -120.

Hence, $x-\frac{3}{2}$, or $2x-3$, is a divisor of the given quantity. Dividing it by $2x-3$, we get the quotient $6x^2-11x-10$, whose factors are $2x-5$ and $3x+2$. Thus,

$12x^3-40x^2+13x+30=(2x-3)(2x-5)(3x+2)$. *Ans.*

25. $12x^3-28x^2+17x-3$.

Note. For convenience of reference we complete the subject of factoring with an article on multinomials. The factoring of such quantities is not often called for, even quadrinomials to be factored appearing much less often than binomials and trinomials. If thought desirable § 82 can be omitted, and referred to when needed. Only the last two exercises of § 85 are multinomials.

5. Multinomials*

82. The Factoring of Multinomials. A great number of kinds o problems might be given, but only six classes will be considered

I. *Multinomials that are the difference of two squares, solved b converse of Theorem* III. See Ex. 1, below.

II. *Multinomials solved by placing sets of terms in parentheses and factoring these parentheses quantities, thus making apparent compound factor.* See Ex. 5.

III. *Multinomials separable into two trinomial factors solved b the trial method.* See Ex. 8.

IV. *Multinomials that are the square of a polynomial, solved b extracting the square root of the square terms, and squaring th sum of the roots by Theorem VI.* See Ex. 12.

V. *Multinomials that are the power of a binomial, solved by extracting the root of the first and last terms and raising the sum o difference of roots to power indicated using Newton's theorem,* § 166.

VI. *Multinomials solved by the divisibility theorem.*

Factor the following multinomials:

1. $a^2 + 2ab + b^2 - c^2 + 2cd - d^2$.

Solution.

$$(a^2 + 2ab + b^2) - (c^2 - 2cd + d^2).$$
$$(a + b)^2 - (c - d)^2.$$
$$(\overline{a + b} + \overline{c - d})(\overline{a + b} - \overline{c - d}). \quad \text{(Theorem III.)}$$
$$(a + b + c - d)(a + b - c + d).$$

Prove by actual multiplication.

2. $4m^2 - 12mn + 9n^2 - p^2 - 4pq - 4q^2$.
3. $(a + b + c)^2 - (d + e + f)^2$.
4. $(a^2 + b^2 - c^2 - d^2)^2 - [2(ab + cd)]^2$.

Separate each of the *factors* of Ex. 4 into two factors.

5. $4x^2 - 16xy + 15y^2 - 4xz + 6yz$.

Suggestion. Put first three terms in a parenthesis and last two in another, and next factor each parenthesis quantity; then proceed as in § 77,

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* See note at end of p. 57.

6. $ax^2 + ax - 6a + x^2 + 7x + 12.$

7. $a^2(b - c) + b^2(c - a) + c^2(a - b).$

SUGGESTION. $a^2(b - c) - a(b^2 - c^2) + bc(b - c).$

8. $6x^2 - 19xy + 15y^2 + 7xz - 13yz - 20z^2.$

SOLUTION.

$$\begin{array}{rrrrrr}
2x & -\,3y & +\,5z & & & \\
3x & -\,5y & -\,4z & & & \\
\hline
6x^2 & -\,9xy & +\,15xz & & & \\
 & -\,10xy & & +\,15y^2 & -\,25yz & \\
 & & -\,8xz & & +\,12yz & -\,20z^2. \\
\hline
6x^2 & -\,19xy & +\,7xz & +\,15y^2 & -\,13yz & -\,20z^2.
\end{array}$$

EXPLANATION. The trial method is used. Coefficients are used which will give $6x^2$, $15y^2$, and $-20z^2$. Then, the two factors chosen are multiplied together. If the given polynomial results, the factors chosen are the ones sought; if it is not obtained, other combinations of coefficients and signs must be tried.

9. $6x^2 - 11xy + 3y^2 - 5xz + 11yz - 4z^2.$

10. $2a^2 + 6ax - 18a + 4x^2 - 24x + 36.$

11. $2a^2 - 4ab - 4ac + 2b^2 + 4bc + 2c^2.$

12. $a^2 + 9b^2 + 25c^2 - 6ab + 10ac - 30bc.$

SOLUTION. $\sqrt{a^2} = a$; $\sqrt{9b^2} = 3b$; $\sqrt{25c^2} = 5c.$
Then test $(a - 3b + 5c)^2$ by § 68.

13. $a^2 + 4b^2 + 9c^2 + 4ab - 6ac - 12bc.$

14. $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4.$

83. The Different Methods used in Factoring.

1. *Method of taking monomial out of polynomial.*
2. *By use of a theorem.*
3. *By trial.*
4. *Method which takes monomials out of parentheses sets.*
5. *Arranging as the difference of two squares and factoring.*
6. *Method for powers of binomials or squares of polynomials.*
7. *Method which uses divisibility theorem.*

III. OUTLINE FOR FACTORING

84. Outline for Factoring with Type Forms.

I. MONOMIALS IN POLYNOMIALS.

Take monomial factors out of all expressions *first*.

$$ab + ac = a(b + c);\quad ax + bx - cx = (a + b - c)x.$$

II. BINOMIALS.

1. *Difference of two squares.*

$$a^2 - b^2 = (a + b)(a - b).$$

2. *Difference of same powers.*

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$
$$a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3).$$
$$a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4).$$

. . . .

a. To get prime factors, factor first by Case 1, whenever it can be used to factor the given quantity or any of its factors.

3. *Sum of same odd powers.*

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$
$$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4).$$

. . . .

4. *Binomials changed to quadrinomials,* the difference of two squares.

$$a^4 + 4b^4 = (a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2.$$
$$= (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2).$$

III. TRINOMIALS.

1. *Trinomial squares.*

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

2. *Trinomials factored by trial method.*

3. *By changing trinomial to quadrinomial.*

$$a^4 + a^2b^2 + b^4 = (a^4 + 2a^2b^2 + b^4) - a^2b^2.$$
$$= (a^2 + ab + b^2)(a^2 - ab + b^2).$$

4. *By divisibility theorem.*

IV. QUADRINOMIALS.

1. *The product of two binomials.*

$$ab + ac + bd + cd = a(b + c) + d(b + c)$$
$$= (a + d)(b + c).$$

2. *The difference of two squares.*

$$a^2 + 2ab + b^2 - c^2 = (\overline{a + b} + c)(\overline{a + b} - c)$$
$$a^2 - b^2 + 2bc - c^2 = a^2 - (b^2 - 2bc + c^2)$$
$$= (a + \overline{b - c})(a - \overline{b - c}).$$

3. *The cube of a binomial.*

$$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3.$$

4. *Divisibility theorem.*

V. MULTINOMIALS.

1. *Problems solved like 1 in quadrinomials.*

2. *Difference of two squares like 2 in quadrinomials.*

3. *Solved by trial method like 2 in trinomials.*

4. *Square of a polynomial.*

$$a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = (a + b + c)^2.$$

. . . .

5. *Powers of binomials.*

$$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 = (a - b)^4.$$

. . . .

6. *By divisibility theorem.*

85. Promiscuous Exercise in Factoring. — In factoring the following exercises, *first set aside monomial factors,* if any are present. Then factor by one of the methods of either of the two preceding articles. If difficulty is experienced, note under which class, whether a binomial, trinomial, etc., the given quantity belongs, and then look for a corresponding type of problem under the corresponding class, and follow the model solution for this type.

Separate each quantity into its *prime* factors, and prove the answers right by a theorem of multiplication or by actual multiplication, if any doubt exists as to the correctness of the result.

1. $a^2 - 25\,b^2$.
2. $a^2 - 12\,a + 36$.
3. $20\,a^3x^3 - 45\,axy^2$.
4. $3\,x^2 + 8\,x + 4$.
5. $27\,n^3 - 8\,p^3$.
6. $x^3 + x^2 + x + 1$.
7. $H^2 - 3\,H + 2$.
8. $x^4 - 13\,x^2 + 36$.
9. $4\,t^2 + t - 14$.
10. $6\,x^3 - 7\,ax^2 - 20\,a^2x$.
11. $x^4 - 7\,x^2 - 18$.
12. $m^6 + 7\,m^3 - 8$.
13. $1 - a^4$.
14. $mnpq + 2 + pq + 2\,mn$.
15. $\pi R^2 - \pi r^2$.
16. $x^2 + (a + b)x + ab$.
17. $x^{16} - y^{16}$.
18. $8\,x^3 - 24\,x^2 + 25$.
19. $8\,a^2 - 2\,a - 3$.
20. $2\,x^4 + x^2 - 28$.
21. $8\,c^2 - 6\,cd - 5\,d^2$.
22. $18\,x^2 + 33\,axy + 14\,a^2y^2$.
23. $\frac{1}{2}\,ab + \frac{1}{2}\,ac$.
24. $\frac{1}{2}\,ab + \frac{1}{2}\,ab' + \frac{1}{2}\,ab''$.
25. $\frac{4}{3}\,\pi R^3 - \frac{4}{3}\,\pi r^3$.
26. $bk - ck + dk + ek$.
27. $20 - 9\,x - 20\,x^2$.
28. $m^3x + m^3y - n^3x - n^3y$.
29. $x^4 + x^2 + 1$.
30. $my^4 + 16\,mx^4 - 12\,mx^2y^2$.
31. $2\,am^5 + 2\,an^5$.
32. $x^9 - 64\,x^3 - x^6 + 64$.
33. $(a + b)^2 - c^2$.
34. $(x - y)^2 - 14(x - y) + 40$.
35. $\frac{1}{6}\,\pi d^3 + \frac{1}{6}\,\pi D^3$.
36. $2\,\pi r^2 + 2\,\pi rh$.
37. $\frac{wty}{2} - \frac{wtr}{6}$.
38. $a(a - c) - b(b - c)$.
39. $\frac{2\,\pi rh}{3} + \frac{2\,\pi r^2}{4}$.
40. $(x^2 + y^2 - z^2)^2 - 4\,x^2y^2$.

41. $125\,a + 64\,ax^{3n}$.
42. $2\,ab - a^2 - b^2 + 1$.
43. $x^{2m} - (y - z)^{2m}$.
44. $21\,x^3 + 26\,x^2 + 25$.
45. $2\,ma^{6m} - 2\,mb^{6m}$.
46. $x^3 - 6\,x^2 + 6\,x - 1$.
47. $9\,x^3 + 5\,x + 50$.
48. $1 + (b - a^2)x^2 - abx^3$.
49. $a^2b^2 - a^2 - b^2 + 1$.
50. $1 - (x - y)^3$.
51. $x^4 - 7\,x^2y^2 + y^4$.
52. $x^3 + y^3 + 3\,xy(x + y)$.
53. $a^2y^3 - b^2x^2y - a^2dy + b^2dx^2$.
54. $x^3 + x^2 - 22\,x - 40$.
55. $6\,x^3 - 3\,x^2 - 33\,x - 6$.
56. $a^{5n} - b^{5n}$.
57. $A^2 - B^2 + C^2 - 2\,AC$.
58. $6\,m^4 - 384\,n^4$.
59. $8\,a^3 + 6\,a^2y^2 - 9\,ay^4 - 27\,y^6$.
60. $125\,x^{15} - 300\,x^{14} + 240\,x^{13} - 64\,x^{12}$.
61. $4\,a^2b^2 - 169\,c^2 + 6\,abd + 39\,cd$.
62. $4\,A^2 - 9\,B^2C^2 + 30\,BCD - 25\,D^2$.
63. $2\,a^2 + 6\,ax - 18\,a + 4\,x^2 - 24\,x + 36$.
64. $2\,x^2 - xy - 3\,y^2 - 5\,yz - 2\,z^2$.

86. Prime Quantities. All the quantities given in the preceding articles were factorable, but quantities *taken at random* are more likely to be prime than composite.

Thus, $3\,x^2 + 7\,x + 9$, $3\,x^2 + 7\,xy - 9\,xz + z^2$, $9\,x^2 - 11\,y^2$, etc., are all prime quantities in terms of rationals. We have seen that the sum of two squares is almost always prime. The expressions $x^2 + xy + y^2$, $x^2 - xy + y^2$, are prime.

It must be remembered, however, as already stated, if fractional exponents are allowed, some of these prime quantities may be factored. See Chapter VII on Fractional Exponent Quantities.

IV. APPLICATIONS OF FACTORING

87. Applications of Factoring. Factoring can be used in the solution of quadratic equations, in arithmetic, and in geometry. It is employed more or less throughout the study of algebra.

88. Application of Factoring in Arithmetic.

1. If a is the amount, b the base, and r the rate in percentage, $a = p + pr$. Factor the right member and state the result in words.

2. To make a rule for squaring any number increased by $\frac{1}{2}$, let a = the number.

Then, $(a+\frac{1}{2})^2 = a^2 + a + \frac{1}{4}$. (By Theorem I.)

$= a(a+1) + \frac{1}{4}$. (By factoring a out of first two terms.)

Hence to square any number plus $\frac{1}{2}$, *multiply the number by one more than itself and add* $\frac{1}{4}$.

Square by this rule the following: $6\frac{1}{2}$, $11\frac{1}{2}$, $25\frac{1}{2}$, 100.5, 7.5.

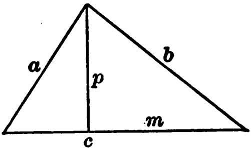
3. Calculate $2 \cdot 3^2 \cdot 17 + 2 \cdot 3^2 \cdot 13 - 2 \cdot 3^2 \cdot 9$.

Suggestion. Take $2 \cdot 3^2$ out first.

4. Calculate $2^2 \cdot 5^3 \cdot 7^3 + 2 \cdot 5^4 \cdot 7^3 - 2^2 \cdot 5^4 \cdot 7^2$.

89. Factoring applied in Geometry.

1. Derive the formula for the area of a triangle when the three sides are given.



Given the triangle whose sides are a, b, c, to find its area.

Referring to result under Ex. 1, § 75, we have

$$a^2 = b^2 + c^2 - 2\,cm,$$

whence $m = \dfrac{b^2 + c^2 - a^2}{2\,c}$. (By transposing both a^2 and $2\,cm$, and then dividing both sides of equation by $2\,c$.)

To find the area of the triangle, we reason now as follows: The area of the triangle is evidently $\frac{1}{2}\,cp$. We know c, and we can find p by using the value of m just found as one side of the right triangle whose three sides are m, p, and b.. Thus,

$$p^2 = b^2 - m^2 = b^2 - \left(\frac{b^2 + c^2 - a^2}{2\,c}\right)^2.$$ (By substituting its value for m.)

$$p^2 = \left(b + \frac{b^2 + c^2 - a^2}{2\,c}\right)\left(b - \frac{b^2 + c^2 - a^2}{2\,c}\right).$$ (Converse Th. III.)

$$p^2 = \left(\frac{2\,bc + b^2 + c^2 - a^2}{2\,c}\right)\left(\frac{2\,bc - b^2 - c^2 + a^2}{2\,c}\right).$$ (Reducing mixed quantities to improper fractions.)

$$p^2 = \frac{(b^2 + 2\,bc + c^2) - a^2}{2\,c} \times \frac{a^2 - (b^2 - 2\,bc + c^2)}{2\,c}.$$ (Arranging terms.)

$$\frac{c^2p^2}{4} = \frac{(b + c + a)}{2}\,\frac{(b + c - a)}{2} \times \frac{(a - b + c)}{2}\,\frac{(a + b - c)}{2}.$$

(§ 81, Exs. 8, 13 and multiplying both sides by $\frac{c^2}{4}$.)

Now let $s = \frac{a + b + c}{2}$; then $s - a = \frac{b + c - a}{2}$; (By Sub. Ax.)

$$s - b = \frac{a - b + c}{2};\quad s - c = \frac{a + b - c}{2}.$$

Then, $\frac{c^2p^2}{4} = s(s - a)(s - b)(s - c)$. (By substituting in the preceding expression.)

$\therefore\ \frac{cp}{2} = \sqrt{s(s - a)(s - b)(s - c)}$. (By extracting the square root of the two equals, Root Axiom.)

But $\frac{cp}{2}$ is the area of the triangle sought. Therefore the right member is equal to the area of the triangle. Notice that we know s when the three sides are given, it being their half sum. The letter s is taken to denote the *half* sum instead of the sum itself, because the formula just found comes out without fractions by so doing. This formula for the area of a triangle is one of the most symmetrical and beautiful in the whole range of mathematics. Its derivation, as we have just seen, is largely a matter of factoring. Exercise in the use of this formula will be given later.

90. Application of Factoring to the Solution of Quadratic Equations. The **degree** of an integral equation equals the exponent of the highest power of its unknown quantity. **Simple** equations are those of the first degree, and **quadratic equations** are of the second degree (§ 18).

1. Solve the equation $x^2 - 4\,x = 32$ and verify.

SOLUTION

$x^2 - 4\,x - 32 = 0$. (Right member made 0 by transposing 32, by Sub. Ax.)

$(x - 8)(x + 4) = 0$. (Factoring left member.)

$x - 8 = 0,\ x + 4 = 0$. (Setting *each* factor equal to zero.)

$\therefore\ x = 8$, *Ans.* $x = -4$, *Ans.* (Transposing known terms to right members, Sub. Ax.)

VERIFICATION. $8^2 - 4 \times 8 \equiv 32.$ $(-4)^2 - 4(-4) \equiv 32.$

Solve the following and verify both roots:

2. $x^2 + 6x = 27.$

3. $3x^2 + 2x - 85 = 0.$

4. $5x^2 - 10x = x^2 - 170x - 700.$

5. $6x^2 + 4x = 3x^2 + 12x + 3.$

6. Find three consecutive numbers such that the sum of their squares shall be 434.

7. The perimeter of a rectangular lot is 90 ft. and its area 450 sq. ft. Find its dimensions.

8. A rectangular garden is 12 rods longer than wide and contains 1 acre. Find its dimensions.

91. Solution of Quadratics by Completing the Square.

1. Solve $x^2 + 6x - 50 = 0$, by completing the square.

SOLUTION

$x^2 + 6x = 50.$ (Sub. Ax., known quantity to right member.)

$6x \div 2\sqrt{x^2} = 3;\ 3^2 = 9.$ (See explanation below.)

$x^2 + 6x + 9 = 59.$ (Add. Ax., adding 9 to both sides.)

$x + 3 = \sqrt{59}.$ (Root Ax., extracting square roots of equals.)

$x = \sqrt{59} - 3.$ (Sub. Ax.)

EXPLANATION. This problem cannot be solved by the factoring method. Hence we turn to the method by completing the square.

By transposing the known term to the right member, we have remaining on the left side a binomial. This cannot be a square, since the square of a monomial is a monomial, and the square of a binomial is a trinomial. We must change the left member into a trinomial square by adding the proper term.

Let us consider that our problem is like having $a^2 + 2ab$ of $a^2 + 2ab + b^2$ given, to find b^2. To get b^2 we divide the second term $2ab$ by twice the square root of a^2, and square the quotient.

2. $x^2 + 12x - 40 = 0.$

3. $4x^2 - 12x = 39.$

a. By means of the completing the square process prime quantities can be separated into factors containing radicals, as will be made clear in a later chapter.

CHAPTER III

HIGHEST COMMON FACTOR — LOWEST COMMON MULTIPLE — FRACTIONS

I. HIGHEST COMMON FACTOR

92. A **common factor** of two or more quantities is a factor that appears in each of them. The **highest common factor** (h. c. f.) of two or more quantities is the product of all the prime factors common to each.

93. Finding Highest Common Factors by Factoring.

1. Find the highest common factor of $6\,a^2b$, $9\,a^3b$, and $24\,a^4b^3$.

Solution. By inspection we see that $3\,a^2b$ is the h. c. f., since any quantity with a greater coefficient or higher exponents would not be contained in each, as $6\,a^3b^2$. The quantities ab and $3\,ab$, though common factors, are not the highest, a^2 being higher than a.

2. $6\,xy^2$, $8\,x^2y$, $12\,x^2y^2$.
3. $9\,m^3n$, $12mn^3$, $15\,m^3n^4$.
4. $7\,a^2b^2$, $21\,a^3b^3$, $14\,a^2b^4$.
5. $8\,a^mb^n$, $12\,a^{2m}b^{3n}$, $15\,a^mb^{4n}$.
6. x^2-y^2, x^3-y^3, $x^2-7\,xy+6\,y^2$.

Solution.

$$\begin{array}{r@{\;=\;}l}
x^2-y^2 & (x+y)(x-y).\\
x^3-y^3 & (x-y)(x^2+xy+y^2).\\
\underline{x^2-7\,xy+6\,y^2} & \underline{(x-6\,y)(x-y).}\\
\text{h.c.f.} & x-y.\quad \textit{Ans.}
\end{array}$$

7. $x^2-2\,x-3$, x^2+x-12, x^2-x-6.
8. $2\,a^3-2\,ab^2$, $4\,b(a+b)^2$.
9. $c^2x^2-d^2$, $acx^2-bcx+adx-bd$.
10. $2\,x^2-9\,x+4$, $2\,x^2-11\,x+5$, $2\,x^2+3\,x-2$.
11. $8\,a^3+1$, $16\,a^4+4\,a^2+1$.

*** 94. Finding Highest Common Factors by Continued Division.** T explain this process two simple principles are needed:

A. *A divisor of a quantity is a divisor of any number of time that quantity.*

Thus, if 6 is contained in 18, it is contained in any integra number of times 18; if $3a$ is contained in $9a^2$ it is contained i $3b \times 9a^2$, or $27a^2b$.

B. *A divisor of each of two quantities is a divisor of their su or difference.*

Thus, if 6 is contained in 66 and in 42 it is contained in thei sum, 108, or in their difference, 24. It is evidently contained i their sum or difference the sum or difference of the numbers o times it is contained in each.

1. Find the greatest common divisor of 258 and 731 by continued division, and justify the process.

SOLUTION

$$\begin{array}{l} 258)731(2 \\ \quad\ \ \underline{516} \\ \quad\ \ 215)258(1 \\ \qquad\quad\ \ \underline{215} \\ \qquad\quad\ \ \ 43)215(5 \\ \qquad\qquad\quad\ \underline{215} \end{array}$$

EXPLANATION OF PROCESS. 731 is divided by 258, which is contained twice with a remainder, 215 Then the *divisor*, 258, is divided by the *remainder* and so on, the divisor each time becoming the new dividend, and the remainder becoming the divisor. The first divisor which is exactly contained in its dividend is the h. c. f. sought.

JUSTIFICATION OF PROCESS. (1) Is any divisor o 258 a divisor of 516? By which principle? Is any common divisor of 731 and 516 a divisor of 215? By which principle? Does it follow then that any common divisor of 258 and 731 must be contained i 215?

(2) Again, is any common divisor of 215 and 516 a divisor of 731? By which principle? Does it follow then that any common divisor of 215 and 258 is contained in 731? Why?

Would it be allowable, then, to drop 731 entirely, and proceed with 215 and 258? Also, in turn, to drop the second dividend 258 for the same reason?

(3) Was it shown at first that the h. c. f. of 258 and 731 could not b greater than 215? Then might it be shown in the same way that it could not be greater than 43?

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* This topic can be omitted if desired, as few problems depend on it, and that they depend on it will be indicated in the text.

(4) Is 43 contained in 215? In 258, the sum of 215 and 43? By what principle? Is 43 contained in 516? By what principle? In 731? By what principle?

Thus, it is shown that the h. c. f. cannot be greater than 43, and that 43 s contained exactly in 258 and 731. Hence 43 is the h. c. f. sought.

Find the h. c. f. in the following by continued division:

2. 2945 and 3441. **3.** 2948 and 8576. **4.** 1911 and 1617.

5. 3094, 4420, 2652, 4662.

SUGGESTION. Find h. c. f. of first two; then find h. c. f. of result and third number; then find h. c. f. of last result and fourth number.

6. *Plan to shorten process.* In practice the process just explained can be materially shortened by removing factors from the irst remainder which are evidently not common to it and either of the given numbers.

For example, in the solution to Ex. 1, the first remainder, 215, evidently contains the factor 5 while 258 does not. Hence 5 can be taken out of 215 and disregarded. Thus 43, instead of 215, becomes the divisor of 258, and is found immediately to be the h. c. f.

Again, in Ex. 2, the first remainder is 496, which contains the factor 2^4. But 2945 does not contain 2. Hence the factor 16 can be taken out of 496, leaving 31, which is then found to be the h. c. f. by dividing 2945 by it.

7. *Easy way to reduce a fraction* having large terms to its owest terms.

Since a fraction is evidently reduced to its lowest terms by dividing numerator and denominator by the h. c. f., we have this practical rule for reducing a proper fraction to its lowest terms:

Multiply the numerator by a number which will make the product approach nearest to the denominator and find the difference between this product and the denominator. Use the remainder to find divisors of both numerator and denominator.

Reduce the following fractions to their lowest terms, or show that any fraction is already in its lowest terms:

8. $\frac{259}{481}$. **Solution.** $259 \times 2 - 481 = 37.$ $\frac{259 \div 37}{481 \div 37} = \frac{7}{13}.$ *Ans.*

9. $\frac{636}{2173}$. 10. $\frac{425}{1207}$. 11. $\frac{619}{1143}$. 12. $\frac{37}{159}$

13. Find the h. c. f. of $x^2 - 6x + 8$ and $4x^3 - 21x^2 + 15x + 2$

Solution.

$$\begin{array}{rl}
4x^3 - 21x^2 + 15x + 20 & \underline{|\ x^2 - 6x + 8} \\
\underline{4x^3 - 24x^2 + 32x} & \underline{|\ 4x + 3} \\
3x^2 - 17x + 20 & \\
\underline{3x^2 - 18x + 24} &
\end{array}$$

$$\begin{array}{rl}
\textit{Ans.}\ \text{h.c.f.} = \underline{x - 4}\ | & x^2 - 6x + 8\ \underline{|\ x - 2} \\
& \underline{x^2 - 4x} \\
& \quad -2x + 8 \\
& \quad \underline{-2x + 8}
\end{array}$$

14. Find h. c. f. of $2x^2 + 11x + 5$ and $2x^2 + 9x + 4$.

15. Find h. c. f. of $6x^3 - 7ax^2 - 20a^2x$ and $3x^2 + 7ax + 4a^2$.

Solution.

$$\begin{array}{rl}
6x^3 - 7ax^2 - 20a^2x & \underline{|\ 3x^3 + 7ax + 4a^2} \\
\underline{6x^3 + 14ax^3 + 8a^2x} & \underline{|\ 2x - 7a} \\
-21ax^2 - 28a^2x & \\
\underline{-21ax^2 - 49a^2x - 28a^3} & \\
7a^2\ \underline{|\ 21a^2x + 28a^3} & \\
3x^2 + 7ax + 4a^2\ \underline{|\ 3x + 4a} & \text{h. c. f.}\quad \textit{Ans.} \\
\underline{3x^2 + 4ax} \qquad \underline{|\ x + a} & \\
3ax + 4a^2 & \\
\underline{3ax + 4a^2} &
\end{array}$$

Explanation. Here we take out the factor $7a^2$ from the remainder a discard it because $7a^2$ is not a factor of one of the given quantities, just a 5 could be discarded in Ex. 1. The old divisor is then divided by the othe factor $3x + 4a$, which is found to be the h. c. f.

16. $6x^2 - 25x + 14$ and $2x^2 - 15x + 28$.

17. $x^3 + 7x^2 - x - 7$ and $x^3 + 5x^2 - x - 5$.

18. $7x^2 + 25x - 12$ and $17x^2 + 67x - 4$.

19. $2x^4 - 7x^3 + 5x^2$ and $x^3 + 3x^2 - 4x$.

Suggestion. First remove monomials from each quantity, reserving th common factor x as a factor of the h. c. f., the other factor being found b continued division.

20. $6x^4 - x^3 - 3x^2 - 4x - 4$ and $8x^4 - 2x^3 - 19x^2 + 3x + 10$.

II. LOWEST COMMON MULTIPLE

95. A common multiple of two or more integral quantities is an integral quantity that is exactly divisible by each of them. The **lowest common multiple** (l. c. m.) of two or more integral quantities is that quantity of smallest coefficient and lowest degree which is exactly divisible by each of them.

For example, a common multiple of $2m$, $3mn$, $6mn^2$ and $9n^3$ is $36m^2n^4$, while their l. c. m. is $18mn^3$.

Find the l. c. m. in the following and test each result to see if it contains each of the quantities and is of the lowest degree possible:

1. $4a^2$, $6a^3b$, $8ab^2$. *Ans.* $24a^3b^2$.
2. $3m$, $2n$, $5m^3n$, $4n^2$. 3. $3a^2bc$, $27a^3b^2c^2$, $6ab^2c$.
4. $56a^2b^3$, $84bx^5$, $48x^4y^6$. 5. a^2-16, a^2-2a-8.
6. a^3+a^2b, a^2-ab, a^2-b^2. 7. $2ab^2c^3x^4$, $3a^4bc^2x^2$, $11a^3bcx$.
8. 60, 120, 48, 36.

Solution. Anything that will contain 120 will contain 60.

$120 = 2^3 \times 3 \times 5$
$48 = 2^4 \times 3$
$36 = 2^2 \times 3^2$
L. C. M. $= 2^4 \times 3^2 \times 5 = 720$. *Ans.*

Notice the *highest* power only of each factor is taken for the l. c. m.

9. 54, 81, 24, 27. 10. 40, 72, 96, 128.
11. x^2-4y^2, $x^2+4xy+4y^2$. 12. x^2+2x, x^2+3x+2.
13. Make a rule for finding lowest common multiple.
14. x^2+5x+6, x^2+6x+8. 15. $x^3(x-y)^2$, $y^3(x^2-y^2)$.
16. $3x^2-10xy+3y^2$, $3x^2-4xy+y^2$.
17. x^2-1, x^3+x^2+x+1, x^3-x^2+x-1.
18. $6x^2-13x+6$, $6x^2+5x-6$, $9x^2-4$.
19. $3x^4+26x^3+35x^2$, $6x^2+38x-28$, $27x^4+27x^2-30x$.
20. $a^2x^2+a^2xy-abx^2-abxy$, $x^3y+2x^2y^2+xy^3$, $3a^3b-6a^2b^2+3ab^3$.

a. In some problems it is necessary to find the highest common factor by continued division. It is then easy to write down the lowest common multiple.

III. FRACTIONS

96. A **fraction** in algebra is an expressed division. The numerator is the dividend and the denominator is the divisor.

97. Reduction of Fractions. By **reduction** in arithmetic and algebra is meant changing the form of a quantity without altering its value.

98. Fundamental Principle in Fractions. *Both terms of a fraction may be multiplied or divided by the same quantity without altering its value.*

Proof. Let $\frac{a}{b}$ = any fraction, r = its value, and m = any number.

Then, $a = br$. (Since the dividend equals the quotient × divisor, §46.)

also, $ma = mbr$. (By the multiplication Axiom, § 57.)

or, $ma = r \cdot mb$. (By the commutative and associative laws of multiplication, § 39.)

Therefore, $\frac{ma}{mb} = r$. (By the Division Axiom, dividing through by mb.)

Or, $\frac{ma}{mb} = \frac{a}{b}$. (Numbers equal to same number are equal to each other, Axiom 8, § 57.)

It should be observed that since a and b may represent any numbers whatever, whether integral, fractional, ordinary irrational, as $\sqrt{3}$, or other, we *assume* they have a definite quotient, and assign r to take the place of this quotient. When we multiply through by m, we *assume* that the multiplication axiom holds true for all values of m and the quantities into which it is multiplied.

99. Exercise in reducing Fractions to their Lowest Terms. Factor both numerator and denominator, and cancel common factors by the principle just given.

Reduce the following fractions to their lowest terms:

1. $\frac{x^4y^6z^2}{x^6y^2z^2}$. 2. $\frac{12a^7b^2}{15a^2b^3}$. 3. $\frac{4x-16}{x^2-16}$. 4. $\frac{x^5-a^2x^3}{x^4-a^4}$.

5. $\dfrac{x^3-27}{x^2-2x-3}$. 6. $\dfrac{x^2-1}{2xy+2y}$. 7. $\dfrac{a^n}{a^{n+1}}$. 8. $\dfrac{a^2-a-20}{a^2+a-12}$.

In some problems it is an advantage to use the continued division method to find the factor by which to divide both terms.

9. $\dfrac{x^3-9x^2+26x-24}{x^3-10x^2+31x-30}$. 10. $\dfrac{6x^3+13x^2+15x-25}{2x^3+4x^2+4x-10}$.

100. Allowable Changes in the Signs of a Fraction. There are *three signs* connected with every fraction; that of the numerator, that of the denominator, and that of the quotient, which stands before the fraction and is called the sign of the fraction.

Using a and b as heretofore to denote any quantities, whether monomials or polynomials, we have

$$+\left(\frac{+a}{+b}\right)=+\left(\frac{-a}{-b}\right)=-\left(\frac{-a}{+b}\right)=-\left(\frac{+a}{-b}\right),$$

since each of these expressions has the value $\frac{a}{b}$. To understand clearly why this is so one must remember that like signs in dividend and divisor give a positive quotient and unlike signs give a negative quotient, and that a − sign before a parenthesis changes the sign of the quantity within when the parenthesis is removed, and that a + sign does not.

Similarly, $$+\left(\frac{-a}{+b}\right)=-\left(\frac{+a}{+b}\right)=+\left(\frac{+a}{-b}\right)=-\left(\frac{-a}{-b}\right),$$

since each of these expressions is equal to $-\frac{a}{b}$.

By examining the preceding results we learn that:

Changing any two of the signs of a fraction does not alter its value, while changing one or all three changes the value of the fraction.

The student will do well, in changing the sign of the numerator or denominator when they are polynomials, to multiply by -1.

The pupil may now explain the following changes:

1. $-\dfrac{-a+b+c}{d}=\dfrac{a-b-c}{d}$; $-\dfrac{3y-x}{m+n}=\dfrac{x-3y}{m+n}$.

a. Notice that $a - b - c = -1 \times (b + c - a)$. If two polynomials have the same terms but each term with the opposite sign, one polynomial is the negative of the other. Thus $-(3y - x) = x - 3y$.

2. $-\dfrac{a+b-c}{c-b} = \dfrac{a+b-c}{b-c}$; $-\dfrac{c}{d-e+f} = \dfrac{c}{e-d-f}$.

3. $\dfrac{-m+n}{-p-q} = \dfrac{m-n}{p+q}$; $\dfrac{-3x-2y+z}{y+z-x} = \dfrac{3x+2y-z}{x-y-z}$.

101. Signs of Factors of Numerator or Denominator Changed. The signs of an even number of the *factors* of the numerator or denominator can be changed without altering the value of the fraction, while changing the signs of an odd number of such factors changes the sign of the fraction. This is because an even number of -1 factors multiplied or divided gives $+1$, while an odd number gives -1.

The student may explain the following:

1. $\dfrac{abc}{(a-b)(c-b)} = -\dfrac{abc}{(a-b)(b-c)}$; $\dfrac{m}{n-m} = -\dfrac{m}{m-n}$.

2. $\dfrac{(y-x)(n-m)}{(b-a)(d-c)} = \dfrac{(x-y)(m-n)}{(a-b)(c-d)}$;
$\dfrac{(n-m)(q-p)}{d-c} = -\dfrac{(m-n)(p-q)}{(c-d)}$.

The student should notice carefully the difference between a **term** of the numerator or denominator and a **factor** of either. Let him define each word. A term can be removed (destroyed) only by addition or subtraction; a factor can be removed (canceled) only by division of both numerator and denominator by the same quantity.

102. A **mixed quantity** is composed of an integral quantity and a fraction. In algebra the sign + or − always appears between the integral quantity and the fraction, since otherwise the operation of multiplication would be indicated (§ 4). Hence, in algebra, reduction of a mixed quantity to an improper fraction is merely a case in addition of fractions.

103. Addition and Subtraction of Fractions.

Type form: $\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}.$

1. $\frac{2}{3} + \frac{5}{6} + \frac{7}{8} + \frac{15}{16}.$

2. $5\frac{2}{3} + 7\frac{3}{10} - 2\frac{7}{15} + 4\frac{7}{20}.$

3. $\frac{x}{x+1} - \frac{x^2 - 3x + 2}{x^2 - 1} + 3.$ Check with $x = 2.$

Solution. L.c.d. (lowest common denominator) is $x^2 - 1$. The denominator of the integral quantity, 3, is 1 understood.

$$\frac{x(x-1)}{x^2 - 1} - \frac{x^2 - 3x + 2}{x^2 - 1} + \frac{3(x^2 - 1)}{x^2 - 1}.$$

(By dividing l.c.d. by the denominator of each fraction, and then multiplying both terms by the quotient, § 98.)

$$\frac{x^2 - x - (x^2 - 3x + 2) + 3x^2 - 3}{x^2 - 1}.$$

(By indicating the addition of the numerators and placing the sum over the common denominator, § 35.)

$$\frac{3x^2 + 2x - 5}{x^2 - 1} = \frac{(3x + 5)(x - 1)}{x^2 - 1} = \frac{3x + 5}{x + 1}. \quad \textit{Ans.}$$

(Simplifying and reducing to lowest terms.)

Check. Let $x = 2$.

Given quantity $= \frac{x}{x+1} - \frac{x^2 - 3x + 2}{x^2 - 1} + 3 = \frac{2}{3} - \frac{4 - 6 + 2}{4 - 1} + 3 = 3\frac{2}{3}.$

Answer $= \frac{3x + 5}{x + 1} = \frac{3 \times 2 + 5}{2 + 1} = 3\frac{2}{3}.$

a. In checking, values which make any denominator 0 must be avoided. Thus, $x = 1$ could not have been used in preceding problem.

4. $\frac{a + x}{a - x} + \frac{a - x}{a + x} + \frac{4ax}{a^2 - x^2}.$

5. $m - \frac{m^2 + n^2}{m - n} + n.$

6. $\frac{a}{a - 2} - \frac{a - 2}{a + 2} + \frac{3}{4 - a^2}.$

Suggestion. Change last fraction to $-\frac{3}{a^2 - 4}.$ (§ 100.)

7. $a + b - \frac{a^2 - 2ab + b^2}{a + b}.$

8. $\frac{x^2 + ax + a^2}{x^3 - a^3} - \frac{x^2 - ax + a^2}{x^3 + a^3}.$

9. $\frac{2}{x^2 - 3x + 2} + \frac{2}{x^2 - x - 2} - \frac{1}{x^2 - 1}.$

10. $\frac{a}{a - b} + \frac{b}{b - c} + \frac{c}{c - d}.$

104. Reduction of Improper Fractions to Mixed or Integral Quantities. An **improper fraction** in algebra is one that can be reduced to a mixed or integral quantity. A **proper fraction** is one that can not be so reduced. To reduce an improper fraction to a whole or mixed quantity, divide the numerator by the denominator, either by short or long division, and add the remainder, if any, over the divisor to the integral quotient.

1. $\frac{ax + 2x^2}{a + x}$. *Ans.* $x + \frac{x^2}{a + x}$.
2. $\frac{x^2 + 2x - 15}{x - 3}$.
3. $\frac{4x^3 - 2x}{2x^2 - x + 1}$.
4. $\frac{a^5x - 3ax^5}{a^3 - ax^2}$.

105. Multiplication and Division of Fractions.

Type forms: $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$; $\frac{m}{n} \div \frac{p}{q} = \frac{m}{n} \times \frac{q}{p} = \frac{mq}{np}$.

1. $\frac{a^2x - x^3}{a^2 - 121} \times \frac{a + 11}{2ax - 2x^2}$.

SOLUTION. $\frac{\cancel{x}(a + x)\cancel{(a - x)}}{\cancel{(a + 11)}(a - 11)} \times \frac{\cancel{a + 11}}{2\cancel{x}\cancel{(a - x)}} = \frac{a + x}{2(a - 11)}$. *Ans.*

CHECK. Let $a = 2$, $x = 1$.

Then, given quantity $= \frac{3}{-117} \times \frac{13}{2} = -\frac{1}{6}$;

also, answer $= \frac{3}{-18} = -\frac{1}{6}$.

2. $\frac{a^2 - 121}{a^2 - 4} \times \frac{a + 2}{a + 11}$.
3. $\frac{a + x}{a} \times \frac{a - x}{x} \times \frac{a^2 + x^2}{a^2 - x^2}$.
4. $\frac{a^2x - x^3}{a} \times \frac{3a}{2ax - 2x^2}$.
5. $\frac{x - 3}{x + 1} \times \frac{x^2 + 2x + 1}{x^3 - 27}$.
6. $\frac{2a^3 - 2ab^2}{a + 2b} \div \frac{a^2 - b^2}{2a + 4b}$.
7. $\frac{8a^5b^6x^7}{15c^6y^5} \times \frac{5a^4b^3c^2}{6x^8y^3} \times \frac{3c^5y^4}{4a^{13}b^4}$.
8. $\frac{45dx - 9dy}{20abx - 10b^2x} \div \frac{30x - 6y}{20a^2x - 5b^2x}$.
9. $-\frac{x^2 + 5x + 6}{x^2 - 1} \div -\frac{x^2 - 9}{x^2 - 2x - 3}$.

10. $\left(\frac{x^2}{y^3}-\frac{1}{x}\right)\div\left(\frac{x}{y^2}+\frac{1}{y}+\frac{1}{x}\right).$

Suggestion. In all such problems reduce both dividend and divisor to simple fractions by § 103.

11. $\left(x^3-\frac{1}{x^3}\right)\div\left(x-\frac{1}{x}\right).$ 12. $\left(x-\frac{3x}{1+x}\right)\div\left(\frac{x^2-2x}{1-x}\right).$

13. $\left[\frac{x^2-64}{x^2+24x+128}\times\frac{x^2+12x-64}{x^3-64}\right]\div\frac{x^2-16x+64}{x^2+4x+16}.$

14. $\left\{\frac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2}\div\frac{3ab-3b^2}{4}\right\}\div\frac{a^2+ab}{2a-2b}.$

106. Complex Fractions. A complex fraction is a fraction that has a fraction in one or both of its terms. It is essentially a problem in division of fractions, and is simplified by dividing the numerator by the denominator.

1. $\dfrac{\frac{3cy}{5a^2x}}{\frac{5b}{7axy}}.$ *Ans.* $\dfrac{21cy^2}{25ab}.$ 2. $\dfrac{\frac{3a^2}{4b}}{\frac{3a}{7b^2}}.$ 3. $\dfrac{\frac{a+x}{2x}}{\frac{y-a}{x}}.$

4. $\dfrac{x-1+\dfrac{6}{x-6}}{x-2+\dfrac{3}{x-6}}.$

Sol. $$\frac{\frac{x^2-7x+12}{x-6}}{\frac{x^2-8x+15}{x-6}}=\frac{(x-3)(x-4)}{x-6}\times\frac{x-6}{(x-3)(x-5)}$$
$$=\frac{x-4}{x-5}.\quad \textit{Ans.}$$

Check. Let $x=7$. Then given fraction $=\frac{12}{8}$; also, answer $=\frac{3}{2}$.

a. A good plan to simplify many complex fractions is to multiply both terms by the l. c. d. of the denominators.

Thus, both terms of Ex. 4 could have been multiplied by $x-6$, which would have given $\frac{x^2-7x+12}{x^2-8x+15}$ directly.

5. $\dfrac{x+\dfrac{2x}{x-3}}{x-\dfrac{2x}{x-3}}.$ 6. $\dfrac{\frac{1}{x}+\frac{4}{x^2}+\frac{4}{x^3}}{1+\frac{5}{x}+\frac{6}{x^2}}.$

107. Powers of Fractions. To raise a fraction to a power, both of its terms, of course, are raised to this power.

$$1.\quad \left(\frac{x^2-1}{x^2-x}\right)^3.$$

SOLUTION. $$\left(\frac{(x+1)(x-1)}{x(x-1)}\right)^3=\frac{(x+1)^3(x-1)^3}{x^3(x-1)^3}=\frac{(x+1)^3}{x^3}.\quad \textit{Ans.}$$

$$2.\quad \frac{(x^2-5x+6)^3}{(x-2)^2}.\qquad 3.\quad \frac{(x^4-xy^3)^{15}}{(x^3-y^3)^{12}}.\qquad 4.\quad \frac{(x^3-1)^3}{2(x^2+x+1)^2}.$$

108. Some Common Errors in Work with Fractions. Certain errors are made so often and by so many students that it is desirable to describe them and caution against them. Checking serves to find such errors.

1. Never cancel single *terms* in the numerator and denominator of a fraction when either is a polynomial. Only *factors* of the whole numerator or whole denominator can be canceled by the fundamental principle (§ 98). Hence, to simplify a simple fraction, cancel factors in both numerator and denominator.

2. In subtraction of fractions, and later in equations, when a fraction is preceded by —, students very often *forget* to change the signs of all the terms of the numerator in simplifying. Because of the danger of forgetting, the use of parentheses is advised for all who are not expert in performing this operation. Thus,

$$\frac{a}{b}-\frac{c-d}{b}=\frac{a-(c-d)}{b}=\frac{a-c+d}{b}.$$

3. To change the signs of both numerator and denominator, the signs of all the terms of each must be changed.

Thus, $\frac{2m-n}{3p-2q}=\frac{n-2m}{2q-3p}$. This amounts to multiplying both terms by -1, by § 98. See § 100.

4. To multiply a fraction by an integral quantity, multiply the numerator by the integral quantity and place result over the denominator. Do *not* multiply both terms by the integral quantity,

since by the fundamental principle this will leave the multiplicand unaltered. There is much less liability of mistakes if 1 is written for the denominator of integral quantities in all four of the fundamental operations in fractions.

5. In division of fractions both dividend and divisor must be either simple fractions or integral quantities. If they are not such, they must be reduced by performing the operations indicated. (See § 105, 10.)

6. In addition of fractions, after the several fractions are reduced to a common denominator and the numerators are added, the sum *must* be placed over the common denominator. Students are particularly likely to forget this when coming back to fractions after studying equations. In the solution of equations, denominators are dropped, the multiplication axiom having been used to make them disappear.

109. Type Forms in Fractions.

1. $\frac{a}{b}=\frac{ma}{mb}$.
2. $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$.
3. $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$.
4. $a+\frac{b}{c}=\frac{ac+b}{c}$.
5. $a-\frac{b}{c}=\frac{ac-b}{c}$.
6. $\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd}$.
7. $\frac{a}{b}\times c=\frac{ac}{b}$.
8. $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.
9. $\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$.
10. $\dfrac{\frac{a}{b}}{c}=\frac{a}{b}\div\frac{c}{1}=\frac{a}{bc}$.
11. $\dfrac{a}{\frac{b}{c}}=\frac{a}{1}\div\frac{b}{c}=\frac{ac}{b}$.
12. $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.

110. Miscellaneous Exercise in Fractions. See § 108 for points to be kept in mind.

Simplify the following:

1. $\frac{mx-nx}{(a+b)(m-n)}$.
2. $a+x-\frac{4ax-5x^2}{a-x}$.

3. $\dfrac{x^2-5x+6}{x^2-7x+12}.$

4. $\dfrac{1}{D-2}+\dfrac{2}{(D-2)^2}.$

5. $\dfrac{a^2b^2+3ab}{4a^2-1}\div\dfrac{ab+3}{2a+1}.$

6. $\dfrac{3-x}{1-3x}-\dfrac{3+x}{1+3x}-\dfrac{1-16x}{9x^2-1}.$

7. $\left(1-\dfrac{a^4}{b^4}\right)\div\dfrac{a^2+b^2}{ab}.$

8. $\dfrac{2}{x}-\dfrac{3}{2x-1}-\dfrac{2x-3}{4x^2-1}.$

9. $(1-G^2)\div\dfrac{(1-G)^2}{G}.$

10. $\dfrac{1}{R-1}-\dfrac{1}{R+2}-\dfrac{3}{(R+2)^2}.$

11. $\dfrac{ax-x^2}{(a+x)^2}\div\dfrac{x^2}{a^2-x^2}.$

12. $\left(x^2+\dfrac{1}{x^2}+2\right)\div\left(x+\dfrac{1}{x}\right).$

13. $\dfrac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}+\dfrac{a-b}{c+d}}.$

14. $\dfrac{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}}.$

15. $\dfrac{x^2+xy}{x^2+y^2}\times\dfrac{x^3-y^3}{xy(x+y)}.$

16. $\left(\dfrac{x^2}{a^2}-\dfrac{x}{a}+1\right)\left(\dfrac{x^2}{a^2}+\dfrac{x}{a}+1\right).$

17. $\left(x^4-\dfrac{1}{x^4}\right)\div\left(x+\dfrac{1}{x}\right).$

18. $\dfrac{3}{x-3}-\dfrac{2}{x-4}-\dfrac{x-6}{(x-2)(x-5)}.$

19. $\dfrac{R^2-r^2}{t^2}\times\dfrac{t^2-t}{R^2-2Rr+r^2}.$

20. $\dfrac{a^{x-2}b^{x+1}c^x}{x^{a-b}y^cz^{a-1}}\times\dfrac{x^{a+b}y^bz^3}{a^2b^{x+1}c^3}.$

21. $\dfrac{1}{1+\dfrac{q+1}{n-q}}.$

22. $\dfrac{2a}{a^4+a^2+1}-\dfrac{1}{a^2-a+1}+\dfrac{1}{a^2+a+1}.$

23. $\left(4x^2+14x+\dfrac{98x-27}{2x-7}\right)\left(\dfrac{1}{6}-\dfrac{3x+29}{12x^2+18x+27}\right).$

24. $\dfrac{4a(a^2-x^2)}{3b(c^2-x^2)}\div\left[\dfrac{a^2-ax}{bc+bx}\times\dfrac{(a+x)^2}{(c-x)^2}\right].$

25. $\dfrac{2a}{(x-2a)^2}-\dfrac{x-a}{x^2-5ax+6a^2}+\dfrac{2}{x-3a}.$

26. $\dfrac{1}{b}\left(\dfrac{1}{a-b}-\dfrac{1}{a+2b}\right)-\dfrac{2}{a^2+ab-2b^2}.$

27. $\left[\dfrac{x^4-a^4}{x^2-2ax+a^2}\div\dfrac{a^2+ax}{x-a}\right]\div\dfrac{x^5-a^2x^3}{x^3+a^3}.$

CHAPTER IV

GRAPHS—RATIO AND PROPORTION—FUNCTIONALITY

I. GRAPHS

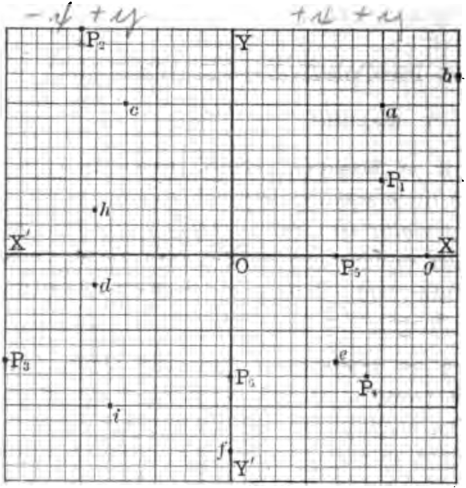
111. Graph Paper. Unit of Measure. Graph paper is paper ery accurately ruled into little squares or parallelograms. Most raph paper has the centimeter (0.4 in. nearly) as the unit of ieasure, and divides each square centimeter into twenty-five ittle squares. Since 10 mm. make 1 cm., one side of the little quare is thus 2 mm. in length.

112. Location of Places by reference to Standard Lines. In raphs the position of a point is located very much as street orners or section corners usually are, as so many blocks or ections north or south, and so many blocks or sections east or est of two streets or roads passing through a central square of town.

113. Axes. Two centimeter lines on the graph paper are made eavier than the others and designated as the *axes*; the horizontal ne as the X axis, and the vertical one as the Y axis. The point here these axes cross is called the *origin*. Other points are cated by saying they are so many units (centimeters or millieters) to the right or left of the Y axis, and so many units above r below the X axis. On the graph diagram distances measured the right from the Y axis are positive, and those to the left are egative; those measured upward from the X axis are positive, nd those downward are negative.

114. Location of Points. The distance to the right or left from ie axis of Y to a point is called its **abscissa,** or x distance; the istance from the X axis to the point is called its **ordinate,** or y istance. The abscissa and ordinate of a point taken together re called the **coördinates** of the point.

1. In the figure, the abscissa of P_1 is $+2$ and the ordinate $+1$ the coördinates of P_2 are $x = -2$, $y = 3$; the coördinates of P_3 ar



$x = -3$, $y = -1.4$; o P_4 are $x = 1.8$, $y = -1.6$; of P_5 are $x = 1.4$, $y = 0$; of P_6 are $x = 0$, $y = -1.6$. For the rules for signs used see the preceding article.

2. Write the coördinates of the points located by the letters a, b, c, d, e, f, g, h, i, on the diagram.

3. On a piece of squared paper draw two heavy lines over

+x −y

two centimeter lines for axes as in the diagram just given. Now locate the following points each by a dot on the diagram, writing a beside the dot of the first point located, b beside the second, and so on.

a.	$(x = 3, y = 1)$.	*b.*	$(x = +3, y = +3)$.
c.	$(x = -2, y = 1)$.	*d.*	$(x = -1, y = 2)$.
e.	$(x = 2.4, y = -1.6)$.	*f.*	$(x = -1.2, y = -1.8)$.
g.	$(x = -.6, y = 2.2)$.	*h.*	$(x = 0, y = -2.4)$.
i.	$(x = -5, y = 0)$.	*j.*	$(x = -4.1, y = -1.6)$.

4. Make another diagram with axes and locate the following points, understanding that the first number inside the parenthesis gives the value of x or the abscissa, and the second number the value of y or the ordinate.

a.	$(2, 2)$.	*b.*	$(2, 6)$.	*c.*	$(-3, 4.6)$.
d.	$(-2, -4.8)$.	*e.*	$(-3, 7.4)$.	*f.*	$(-3, -8.1)$.
g.	$(-4.9, 0)$.	*h.*	$(7.3, -2.6)$.	*i.*	$(0, 0)$.
j.	$(0, -6.5)$.	*k.*	$(12, -14)$.	*l.*	$(5.5, 0)$.

115. Graphs. If a series of points is "plotted" (that is, located and marked on a diagram), representing values of a quantity y that changes as another quantity x changes, and these points are joined by a running line, this line is called the **graph** of the law or data that determined the points.

In § 60 a number of graphs were constructed from statistical data. Such data can come from a great variety of sources. Thus, writers on history and economics use graphs to show to the eye quickly the changes in population, expenditures, and production. Scientists use them to show the laws of nature, engineers to show he working of machinery, and business houses to show the changes in prices, cost of production, sales, etc. In short, graphs have a wide range of uses, and the student should learn to contruct and read them readily. In science and engineering graphs are often constructed mechanically, as by the thermograph, barograph, anemograph, etc.

1. **Statistical Graph.** The table gives the number of survivors t different ages out of 100,000 particular persons alive at age ten:

Age	Survivors	Age	Survivors	Age	Survivors
10	100,000	40	78,106	70	38,569
15	96,285	45	74,173	75	26,287
20	92,637	50	69,804	80	14,474
25	89,032	55	64,563	85	5,485
30	85,441	60	57,917	90	847
35	81,822	65	49,341	95	3

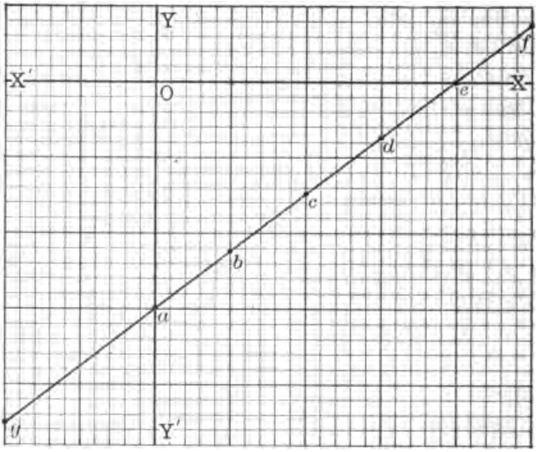
Let 1 cm. on Y axis represent 10,000 persons, and 1 cm. on X xis ten years. Then coördinates of first point are $x = 1$, $y = 10$.

2. **Graph of an Equation.** In the equation, $y = 3x + 5$, we see that y is a function of x, that is, depends on x for its value. As x changes in value, y also changes correspondingly.

We see that for every value we assign to x there will be a correponding value for y. Such sets of corresponding values of x and can be taken for the coördinates of points. If these points are connected with a running line, we have a graph. Evidently any equation, as $y = x^2 + 4x - 6$, will have a graph to correspond to it.

116. Graphs of Simple Equations or Equations of the First Degre (§ 90).

1. Construct the graph of the equation $3x - 4y = 12$.



Table

$3x - 4y = 12$

(x, y)	Pt.
$(0, -3)$	a
$(1, -2.25)$	b
$(2, -1.5)$	c
$(3, -.75)$	d
$(4, 0)$	e
$(5, .75)$	f
$(-2, -4.5)$	g

Explanation o how the values in the table are found. If $x = 0$ in the given equation, $3x - 4y = 12$, the equation reduces t $-4y = 12$, whence $y = -3$, as given in the table opposite a. If $x = 1$, $3x - 4y = 12$ becomes $3 - 4y = 12$, whence, transposing 3, $-4y = 9$, whence, $y = -2.25$, which is given opposite b; and in the same manner the other values of y are found. We simply take any convenient values for x and find the corresponding values of y, and write the pairs of correspondin values in parentheses for the coördinates of points.

Having found the coördinates of a series of points, a, b, c, d, e, f, g, we now locate these points on a diagram. The point a, whose coördinates are $(0, -3)$, is located first by a dot with the letter a beside it; then the point b is located in the same way; and so on. Last of all a line is drawn through these points.

The student may now construct in the same way the graphs of the following equations. He can let $x = 0, 1, 2, 3, 4, 5, -2$ as in the preceding example.

2. $3x - 2y = 5$. **3.** $3x + 5y = 15$. **4.** $2x - 7y = -12$.

5. *The graph of a simple equation, or an equation of the first degree, will be found to be always a straight line,* as in Ex. 1–4.

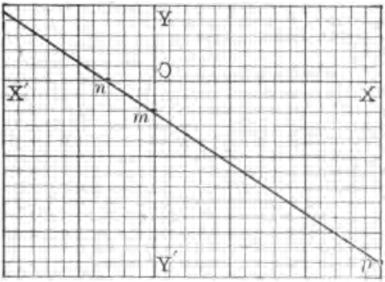
This being the case, since *two* points determine a straight line, ; will be necessary to locate only two points to locate the graph n the diagram. Now the easiest pair of coördinates to obtain are hose found by first putting $x=0$, and getting the corresponding alue of y, and then putting $y=0$, and getting the corresponding alue of x.

To check the correctness of the location of the graph, it is esirable to locate a third point near the margin of the sheet. ow, whenever the two points first located fall close together, a ight inaccuracy in plotting either of them will throw the ex- emities of the graph very considerably out of the correct posi- on. In such instances the graph should be constructed by ining the marginal point to the more remote of the first two. he other point is then used only as a check. Calculate decimals sually to tenths only.

6. Construct the graph of $10x+15y=-6$.

$10x+15y=-6$

(x, y)	PT.
$(0, -.4)$	m
$(-.6, 0)$	n
$(3, -2.4)$	p



Here, because m and n are close ;ether, we join n and p to get the ıph, and check with m.

Construct the graphs of the following equations, following the ›del just given:

7. $5x+3y=-10$.

8. $-2x+5y=12$.

9. $6x-7y=1$.

10. $25x+7y=29$.

11. $6x+5y=16$.

12. $35x+17y=-86$.

13. $x=5$.

SUGGESTION. y can have *any* value. Locate two points. The graph is ne parallel to the Y axis, 5 units to right of YY'.

14. $y = -3$. 15. $\frac{x}{2} - \frac{y}{3} = 4.5$. 16. $\frac{x + 3y}{2} = \frac{y}{5} - 1$.

17. $2x = 5y$.

Suggestion. Putting $x = 0$ or $y = 0$ gives the origin. To find a secon point, another value must be assigned to x or y.

18. $3x - 11y = 0$. 19. $x = y$. 20. $x = -y$.

21. $x + 2y = 75$.

Suggestion. In such a problem the centimeter as unit runs the point located off the diagram. Instead of the centimeter use the millimeter o 2-millimeter length as unit.

22. $3x - y = 120$. 23. $\frac{x}{13} - \frac{y}{7} = 6x - 10y - 8$.

II. RATIO AND PROPORTION

117. The **ratio** of two quantities is the *quotient* obtained by dividing the first by the second. A ratio is thus a fraction It is usually denoted by a colon, but preferably by a fraction Thus, $2:3$ or $\frac{2}{3}$ is the ratio of 2 to 3, the sign : having the same meaning as $\div$.

1. What is the ratio of 9 to 12? Of a to b? Of $a + b$ to $c + d$? Of $a^2 - b^2$ to $(a + b)^2$? Of $\frac{7}{12}$ to $\frac{5}{7}$? Of a to $b + c$?

2. The first number in a ratio is called its **antecedent**, and the second its **consequent**; both are called the **terms** of the ratio. A ratio is always an abstract number, whatever concrete numbers may be in its terms, since it is always the quotient of one number divided by another of the same kind.

118. A **proportion** is an equation (§ 54) whose members are ratios.

For example, $\frac{6}{9} = \frac{10}{15}$; $6:9 = 10:15$; $\frac{a}{b} = \frac{c}{d}$ are proportions.

By definition four terms are in proportion when the first term divided by the second equals the third term divided by the fourth.

119. Fundamental Theorem in Proportion. *In a proportion, the product of the first and last terms, called the "extremes," equals the product of the two middle terms, or "means."*

Proof. Let $\frac{a}{b}=\frac{c}{d}$ be any proportion.

Then, $\frac{a}{b}\times bd=\frac{c}{d}\times bd$, (By the multiplication axiom, § 57.)

or $ad=bc$. (By rule for multiplication of fractions. Notice a and d are the extremes, and b and c, the means.)

The importance of this theorem consists in the fact that it changes a proportion into an *integral equation.*

120. Exercise in Proportion. Using the definition, § 118, to solve Ex. 1, and the theorem of § 119 to solve Exs. 2, 3, find the unknown in the following proportions:

1. $\frac{8}{16}=\frac{36}{x}$; $\frac{x}{96}=\frac{7}{42}$; $100:x=50:75$; $45:60=x:24$.

2. $8:9=10:x$; $\frac{2}{3}:\frac{1}{2}=x:7$; x da. : 7 da. $=21:16$.

3. Solve $\frac{a}{b}=\frac{c}{d}$, or $a:b=c:d$, for a, b, c, and d in turn.

4. The answers to Ex. 3 can be written $a=b\times\frac{c}{d}$; $b=a\times\frac{d}{c}$; $c=d\times\frac{a}{b}$; $d=c\times\frac{b}{a}$. Now, $\frac{a}{b}=\frac{c}{d}$ is a formula for every simple proportion. We see, then, that the unknown is always equal to the term corresponding to it multiplied by the proper ratio of the other two terms. Hence we have the following:

5. **Rule** for solving any arithmetical problem in proportion whose terms are concrete numbers. *Write first the number corresponding to the answer with its name; multiply this by the improper fraction ratio of the remaining two terms if the answer is to be greater than the first factor, and by the proper fraction ratio, if it s to be less.*

Always reduce the ratio factor to its lowest terms before writing it down, and cancel to the best advantage before multiplying.

This is the business man's rule for solving any analysis or proportion problem, and should be mastered by the student. It is he easiest and quickest way available.

Solve the following problems by the rule just given and check the result by an analysis solution.

6. If 45 yd. of cloth cost \$72, what will be the cost of 65 yd.?

SOLUTION. \$72 × $\frac{13}{9}$ = \$104. *Ans.*

EXPLANATION. \$72 corresponds to the answer. Answer is to be more than \$72, so ratio is improper fraction, viz. $\frac{65}{45}$.

CHECK. If 45 yd. cost \$72, 1 yd. costs \$1.60; then 65 yd. cost 65 × \$1.60 or \$104.

7. If a building known to be 75 ft. high casts a shadow 125 ft. long, how high is a tree that casts a shadow 105 ft. long?

8. If an express train travels 375 mi. in 10 hr., what time would it require to go 525 mi.?

9. 26 men build a wall in 21 da.; how many men could have built it in 14 days?

10. A water tower 100 ft. high and 22 ft. in diameter is to be represented in a drawing as 8 in. high. How many inches shall there be in the diameter of the drawing?

11. The height of the mercury in a barometer is proportional to the pressure. If the pressure is 14.7 lb. to the square inch when the mercury is 30 in. high, what is the pressure when the mercury is only 28 in. high?

12. If the wages of 48 men for 11 days of 9 hr. each is \$1276, what would be the wages of 27 men for 13 days of 8 hr. each?

SOLUTION. \$1276 × $\frac{9}{16}$ × $\frac{13}{11}$ × $\frac{8}{9}$ = \$754. *Ans.*

EXPLANATION. Each ratio is found independently of the others. Thus, if a certain sum is earned in 11 da., in 13 da. there will be earned $\frac{13}{11}$ times as much.

CHECK. A "cause and effect" solution is much preferable here to an analysis one as a check. By this method the two causes, or the products of the numbers of men, days, and hours, and the two effects, dollars, form in order the four terms of a simple proportion, x taking the place of the unknown. Evidently a "man-hour," or what a man can do in an hour, is the unit used in the causes.

13. If 6000 ft. of copper wire $\frac{1}{16}$ in. in diameter costs \$10 when copper is 15¢ a pound, what will be the cost of 3850 ft. of $\frac{3}{32}$ in. wire when copper is 14¢ a pound?

Suggestion. The cost will be in proportion to the *areas* of the cross sections, or to the squares of the diameters $\frac{1}{16}$ and $\frac{3}{32}$.

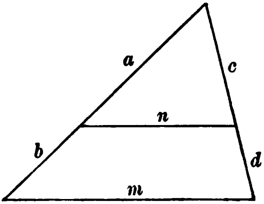
14. Three men go into partnership, the first furnishing $ 6000 f the capital, the second $5000, and the third $4000. They lear the first year $2400. What is each man's share of the rofits ? Show how to solve this problem by the rule in 5 above.

15. A bankrupt owes three creditors $240, $360, and $400 espectively. If $325 is divided among them in proportion to heir claims, what will each one get?

16. If silver solder contains 1 part brass, 1 part copper, and 9 parts silver, how much of each would there be in 20 oz. of older ?

17. A substance by analysis was found to contain 8.08 units (by veight) of sodium, 5.61 units of sulphur, and 11.31 units of oxygen. Iow many pounds of sulphur are in 11.4 lb. of the substance ?

18. If a line n is drawn parallel to the base m of a triangle, diiding the other two sides into the four segments a, b, c, d, make ll the proportions you can out of these ix letters. See § 51. Find two different xpressions for the value of a. Also two or m.



19. If two chords intersect in a circle, nake a proportion out of the two segnents of the first chord respectively, a nd b, and the two segments of the second chord respectively, and d. If $a = 4$, $b = 6$, $d = 11$, find c. See § 51.

20. If two secants a and b intersect without a circle, make a roportion out of them and their external segments which are repectively c and d. Find c when $a = 10$, $b = 12$, $d = 4$.

21. If a perpendicular p is dropped from the vertex of a right riangle on the hypotenuse h, dividing it into the segments m and lying respectively next to the other two sides of the triangle, a nd b, what proportions hold between these letters ? Solve for m

and n, getting two different expressions as values for m. Fin
m when $a = 6.5$ and $h = 25$.

22. If t is a tangent to a circle from a point outside and s is
secant from the same point and e is its external segment, find
when $t = 6.4$ and $s = 13$.

121. Study of Proportion in which two terms are given, the third term has various values *assigned* to it, and the corresponding values of the fourth are to be found.

A typical proportion problem has *two* kinds of quantities involved, each having two values; as, if 5 lb. of nails cost 28¢, what will 7 lb. cost? We readily see that if we know that 5 lb. cost 28¢, we can *take* any other number of pounds and find the cost, or any cost and find the corresponding number of pounds. This truth will be used in the next article.

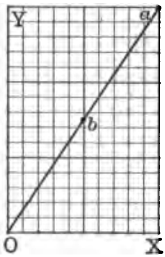
122. Proportions and their Graphs. For a graph we need two varying quantities, x and y. In a proportion there are four terms of which x and y can be two and the other two must be known numbers.

1. To find the graph of the proportion $x : y = 2 : 3$, or $y = \frac{3}{2} x$.

SOLUTION:

$$y = \tfrac{3}{2} x$$

(x, y)	PT.
(0, 0)	0
(2, 3)	a
(1, 1.5)	b



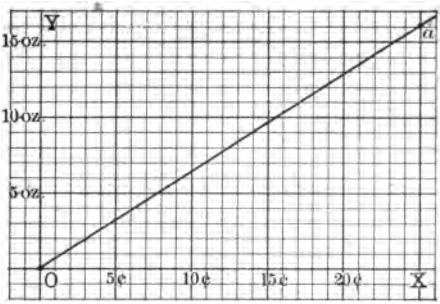
We see from this that the graph of a proportion, of which x and y are the terms of one ratio, is *a straight line through the origin* and through a point whose coördinates are the terms of the given ratio, the term corresponding to x being the *abscissa* and that corresponding to y the ordinate.

Since three terms are always given in a proportion to find the fourth, we must *assign* values to either x or y, and the graph will then give a corresponding value for the other letter. Thus, if $y = 2$ in the graph, the corresponding value of x is found by running along the graph $y = 2$ (or the line parallel to x and 2 units above it) until we come to graph Oa, and thence down vertically to OX, where we get the corresponding value of x, or 1.3^+. Or, again, if $x = .8$, y is found to be 1.2 by passing around in the opposite way.

2. Construct a cost graph when 1 lb., or 16 oz., costs 25¢.

Solution. The graph passes, of course, through the origin, and is made to go through the point (25, 16). From it can be read off the cost of any number of ounces by starting at the number of ounces above O, going thence on a horizontal line to the graph, thence downward in a vertical line to OX, where the cost can be read off from the OX scale of numbers.

Similarly the number of ounces corresponding to any given cost can be found by starting at the cost point on OX, going thence on a vertical line upward to the graph, thence on a horizontal line to OY, where the number of ounces can be read off from the OY scale.



3. Using the graph to Ex. 2, find the cost of 5, 8, 9, 17 oz., respectively, checking each time with an arithmetical solution. Find, also, the number of ounces corresponding to 12, 18, 23, 4 cents, respectively, and check with arithmetical solutions.

4. Solve graphically Ex. 6–9, § 120.

5. Make a graph to reduce from pounds to kilograms, and conversely, knowing that 1 kg. = 2.2 lb. (nearly); also one to reduce from miles to kilometers, and conversely, knowing that 1 mi. = 1.6 km. (nearly); also one to reduce from meters to yards, and conversely, knowing that 1 m. = 1.1 yd. (nearly). (These figures are approximations, correct to one decimal place.)

6. Make a graph to reduce diameters to circumferences, and conversely, knowing that one circumference = $3\frac{1}{7}$ diameters.

7. Construct graphical interest tables for 4 %, 5 %, 6 %, respectively.

Suggestion. Assume \$100 as principal. Then use 360 da. for known x, and \$4 for known y. Let 1 cm. = 60 da. on X axis; 1 cm. = \$1 on Y axis.

8. Construct graphs for reducing from dollars to English pounds; dollars to French francs; dollars to German marks, knowing that £1 = \$4.8665, 1 fr. = 19.3¢, and 1 M. = 23.8¢.

9. Construct a graph for uniform motion when distance 20 mi. is covered in $3\frac{1}{2}$ hr. Also graph when rate is given as so many miles an hour, as 14 mi. an hour. Find from first graph distance covered in $2\frac{3}{4}$ hr. Also time it takes to go 33 mi.

III. FUNCTIONALITY

123. A **constant** is a quantity that retains the same value throughout an investigation. Thus, in § 122, 1, the 2 and 3 are constants.

124. A **variable** is a quantity that changes in value. Thus, in § 122, 1, x and y are variables. The height of the mercury in a thermometer is a variable, as it is always slowly changing, either rising or falling. The speed of a train is a variable, always increasing or diminishing, though at times the change is very slight. The age of a man, the height of a tree, the cost of most articles, are all variables. Indeed, strictly speaking, nearly all quantities in nature are variables, though some of them change so slowly and slightly (as the length of an iron rod) that they are commonly regarded as constants.

Some variables change continuously while others change discontinuously. Thus, the age of a man changes continuously, while the price of an article, as the cost of butter, usually jumps from one integral number of cents to another. In the mathematical treatment such discontinuous variables are regarded as continuous.

Variables are commonly represented by the last letters of the alphabet, and constants by the others, though this rule is often broken.

125. Dependent and Independent Variables. Variables are usually connected, one depending for its value on the value of another.

For example, in a statistical graph which gives the temperatures throughout the hours of a day, we see that *the thermometer reading* depends on *the hour of the day.* In the formula $s = \frac{1}{2}gt^2$, as t increases s also increases as the body falls, both being continuous variables.

Similarly, we saw in the article on functions (§ 53) that any quantity containing x is a variable, if x is a variable.

Thus, $\frac{3x-2}{5}$ varies in value if x varies.

If $x=0$, $\frac{3x-2}{5} = -\frac{2}{5}$; if $x=1$, $\frac{3x-2}{5} = \frac{1}{5}$; if $x=2$, $\frac{3x-2}{5} = \frac{4}{5}$; if $x=3$, $\frac{3x-2}{5} = \frac{7}{5}$; etc.

The common plan is to let the letter y equal the function of x. Thus, in the preceding example, we would let $y = \frac{3x-2}{5}$.

Then, if $x=4$, $y=2$; if $x=5$, $y=2.6$; if $x=10$, $y=5.6$; etc.

It is customary to call x the *independent* variable and y the *dependent* variable.

126. The Use of Graphs to show Functional Relations. Such equations as $y = \frac{3x-1}{4}$ or $y = \frac{-3x-1}{4}$ can have graphs constructed for them as in § 116. Now by first locating a series of points and then drawing a *running line* through them, we see the graph showing a *continuous increase* in the value of x, and a corresponding *continuous increase* or *decrease* in the value of y.

1. Construct the graph for $y = 1 + 3x$, as in § 116.

Here we see that as x increases y also increases, and as x decreases y also decreases. Notice also that corresponding to *any* intermediate value of x between any two points there is a value of y which can be read off very approximately correct.

2. Construct the graph for $y = 2 - 3x$.

In this problem we see that as x *increases*, y *decreases* algebraically, and *vice versa*.

3. A small factory has a daily expense account of \$15 for wages, materials, etc. The average sales per day per workman are \$3. Make a graph from these conditions, using x for number of workmen, and y = number of dollars of net profit.

4. A merchant's fixed charges for rent, interest on borrowed money, clerk hire, etc., are \$4000 annually. His gross profit on sales averages 40 %. Construct a graph in which x = number of thousands of dollars of annual gross sales, and y = number of thousands of dollars of net profit.

From this graph find y when $x = 10, 20, 5$ respectively. We see here how net profits is a function of gross sales.

5. The formula for uniform motion is $s = vt$, in which s = number of units of distance passed over, v = number of units of distance passed over in one unit of time (velocity), and t = number of units of time. For uniform motion v is constant.

The graph of § 122, 9 is of uniform motion, y taking the place of s and x that of t.

6. If the typesetting for a circular costs \$3, and the paper and printing 6 ¢ a copy, construct graph from these data, letting y = total cost, and x = number of copies bought. From graph find cost of 500 copies. Find also number that can be bought for \$8.

7. If the sundry charges for a lecture course (advertising, tickets, etc.) cost \$100, and the talent costs \$1.25 per ticket, construct graph showing net profit or loss for varying numbers of tickets disposed of, if tickets are sold at \$1.50. Suppose 300, 500, 800 tickets are sold; what is gain or loss in each case?

CHAPTER V

SIMPLE EQUATIONS

I. DEFINITIONS

127. An **equation**, as already defined in § 54, is a statement that two quantities have the same numerical value.

128. Equations are of two kinds: equations of condition, and identities.

129. A **conditional equation** is one in which the two sides have the same numerical value only when a particular value, or values, are substituted for the unknown number.

Thus, $3x + 5 = 35$ is a conditional equation, since $x = 10$ is the only value of x which makes the two sides of the equation equal. Again, $x^2 - 5x + 6 = 0$ is "satisfied" only when $x = 2$, *or* when $x = 3$. If we put $x = 4$, the left side, or *member*, of the equation equals 2, reducing the equation to $2 = 0$, which is not true.

By an equation is commonly meant a conditional equation.

130. An **identity** is a statement that a quantity is equal to itself, or that two quantities are the same when both are reduced to their simplest form. The sign for an identity is $\equiv$.

For example, $ax + b \equiv ax + b$; $3 + 4 \equiv 2 + 5$;

$$(x - 4)(x - 5) \equiv x^2 - 9x + 20.$$

Identical equations, unlike conditional ones, hold true for *any* values of the letters contained in them. For instance, $x = 2, 3, 4, -2$, etc., substituted for x in the equation $(x - 4)(x - 5) \equiv x^2 - 9x + 20$ all satisfy it. Test this statement.

The given quantities in exercises in addition, subtraction, multiplication, division, and factoring, in Chapters I, II, III, set equal to the answers give rise to identical equations. The ordinary sign for equals, "$=$," is very often used in place of $\equiv$. An identity is a declarative sentence. (See § 54.)

131. Conditional Equations Classified. Conditional equations are classified:

1. According to the degree (§§ 18, 90) of the unknown quantity, into: simple, quadratic, cubic, quartic, quintic, etc., equations. A **simple equation** is one of the *first degree;* a **quadratic equation** is one of the second degree; a **cubic** one of the third degree; etc. Notice the technical use of the word *simple.* It is easy to forget this definition, if one does not pay special attention to it.

2. According to the number of unknown quantities they contain, into: equations containing *one* unknown, and simultaneous equations.

132. Known and Unknown Quantities. The custom in algebra has been to denote unknown numbers by the last letters of the alphabet and known numbers by the others, but many authors are now using initials for the unknowns in problems. In science the rule is to use initials; as, v for velocity, p for pressure, etc.

133. The **root** of an equation is the value of the unknown number. Contrast root of an equation and root of a quantity (§ 13). To **solve** an equation is literally to untie the unknown number from the other quantities, thus finding its value.

134. To **verify a root** of an equation is to show that this value of the unknown number being substituted for the unknown in the given equation reduces it to an identity.

Before beginning the next article the pupil should **carefully** review §§ 54–58. The axioms should be thoroughly memorized.

II. EQUATIONS CONTAINING ONE UNKNOWN QUANTITY

135. Solution of Equations, especially fractional equations, by neans of the axioms.

Solve and verify in the following:

1. $\dfrac{x-3}{2}-\dfrac{x-2}{3}=\dfrac{3x-21}{12}.$

Solution. L. c. d. $=12$.

$\therefore 6x-18-(4x-8)=3x-21.$ (Mult. Ax. Equals, $\dfrac{x-3}{2}-\dfrac{x-2}{3}$ and $\dfrac{3x-21}{12}$, multiplied by the same number, 12, (the l. c. d.), give the equals, $6x-18-(4x-8)$ and $3x-21$.)

$\therefore 6x-18-4x+8=3x-21.$ (Removing parentheses, § 36.)

$\therefore 6x-4x-3x=+18-8-21.$ (Sub. Ax. The same quantity, $-18+8+3x$, subtracted from equals, $6x-18-4x+8$ and $3x-21$, leaves the equals, $6x-4x-3x$ and $+18-8-21$.)

$\therefore -x=-11.$ (Adding on each side, § 31.)

$\therefore x=11.$ *Ans.* (Div. Ax. Equals, $-x$ and -11, divided by the same quantity, -1, give the equals, x and 11.)

Verification. $\dfrac{11-3}{2}-\dfrac{11-2}{3}=\dfrac{3\times 11-21}{12}$; or, $4-3\equiv 1.$

2. $\dfrac{x-2}{5}-\dfrac{x-3}{4}=\dfrac{x-7}{10}.$

3. $\dfrac{x}{2}-2=\dfrac{x}{4}+\dfrac{x}{5}-1.$

4. $x+\dfrac{11-x}{3}=\dfrac{26-x}{2}.$

5. $\dfrac{3n}{5}+7-n=\dfrac{1}{2}.$

6. $\dfrac{x}{x+2}+\dfrac{4}{x+6}=1.$

7. $3+\dfrac{n}{.5}=10-\dfrac{n}{.2}.$

8. $\dfrac{x+4}{3x-8}-\dfrac{x+5}{3x-7}=0.$

9. $x=3x-\frac{1}{2}(4-x)+\frac{1}{3}.$

10. $\dfrac{7}{t}=\dfrac{23-t}{3t}+\dfrac{t-1}{4t}.$

11. $\dfrac{3n+7}{4n+8}=\dfrac{6n-2}{8n-5}.$

12. $\frac{8}{u}+\frac{u-3}{u+3}=\frac{u+1}{u-1}.$

13. $\frac{4}{x+2}+\frac{7}{x+3}=\frac{37}{x^2+5x+6}.$

14. $\frac{x}{7}=\frac{12x-60}{11}-\left(\frac{2x}{77}+1\right).$

15. $\frac{2x+1}{(x+2)^2}+\frac{2x+1}{x+2}=2.$

16. $\frac{1}{5-\frac{1}{x}}=\frac{2}{7}.$

17. $\frac{3}{4}-\frac{\frac{3}{4}x+\frac{3}{4}}{x+\frac{3}{4}}=\frac{\frac{3}{4}}{x+\frac{3}{4}}-\frac{3}{4}.$

18. $\frac{2x+19}{5x^2-5}-\frac{17}{x^2-1}=\frac{3}{1-x}.$

19. $\frac{1}{4}(4+\frac{3}{2}x)-\frac{1}{7}(2x-\frac{1}{3})=\frac{31}{28}.$

20. $3-\frac{1}{t}=\frac{1}{\frac{1}{3}+\frac{1}{t}}.$

21. $\frac{7y}{y+3}-\frac{5y}{1-y}=\frac{12(y^2-1)}{y^2+2y-3}.$

22. $\frac{1}{5}-\frac{3}{x-1}=\frac{2+\frac{x+4}{1-x}}{3}.$

23. $\frac{1}{3}(s-3)-\frac{1}{4}(s-8)+\frac{1}{5}(s-5)=0.$

24. $(x-1)^3+x^3+(x+1)^3=3x(x^2-1).$

25. $(x+\frac{5}{2})(x-\frac{3}{2})-(x+5)(x-3)+\frac{3}{4}=0.$

26. $(x+1)^2=\{6-(1-x)\}x-2.$ See § 21, 2.

27. $\frac{2z^2-z+3}{3z+2}-\frac{2z^2+3z-1}{3z-2}=\frac{3-6z-20z^2}{9z^2-4}.$

28. $\frac{x^2+3}{2(x^3-8)}+\frac{1}{6(x-2)}=\frac{2x-1}{3(x^2+2x+4)}.$

136. Special Processes of Solution. The regular method of solution is: (1) clear of fractions; (2) transpose; (3) collect; (4) divide through by the coefficient of x: but *any* method is permissible which solves or loosens x from the other quantities, provided the several steps can be justified. We can therefore use this rule: *Proceed in any manner that seems advantageous, by use of the axioms, to get x alone on one side, and only known quantities on the other side of the equation.*

Such irregular processes of solution often possess a distinct advantage over the normal process.

1. $\frac{x-1}{x-2}-\frac{x-5}{x-6}=\frac{x-3}{x-4}-\frac{x-7}{x-8}.$

Solution. $\frac{x-7}{x-8}-\frac{x-5}{x-6}=\frac{x-3}{x-4}-\frac{x-1}{x-2}.$ (Sub. Ax.)

$$\frac{x^2-13x+42-(x^2-13x+40)}{x^2-14x+48}=\frac{x^2-5x+6-(x^2-5x+4)}{x^2-6x+8}, \text{ (Adding.)}$$

or

$$\frac{2}{x^2-14x+48}=\frac{2}{x^2-6x+8}.$$

$$2x^2-12x+16=2x^2-28x+96. \quad \text{(Mult. Ax.)}$$

$$16x=80. \quad \text{(Sub. Ax.)}$$

$$x=5. \quad \text{(Div. Ax.)}$$

Observe that transposing came first and clearing of fractions near the last instead of at first.

2. $\frac{1}{x-2}-\frac{1}{x-4}=\frac{1}{x-6}-\frac{1}{x-8}.$

Suggestion. Add on each side first.

3. $\frac{2}{x-14}-\frac{5}{x-13}=\frac{2}{x-9}-\frac{5}{x-11}.$

Suggestion. Transpose the two terms on either side of =, and then add in each member.

4. $\frac{9x+14}{12}-\frac{1}{7x}=\frac{5}{6x}-\frac{16-21x}{28}.$

Suggestion. Reduce fractions to mixed quantities first, and destroy $\frac{3}{4}x$.

5. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}.$

Suggestion. Use Suggestions to 4 and 2.

6. $\frac{x^2-2x+1}{12}+\frac{5x-5}{24}=\frac{(x-1)^3}{12x}.$

Suggestion. Divide by $x-1$. Then, $x=1$ is a root, § 90.

If denominators contain both monomial and other expressions, it is often best to clear of monomial denominators *only* at first, and then simplify before proceeding to clear of the remaining denominators.

7. $\frac{9x+3}{27}+\frac{3x-6}{2x-5}=\frac{2}{3}+\frac{3x+22}{9}.$

SOLUTION

$9x+3+\frac{81x-162}{2x-5}=18+9x+66.$ (Multiply through by 27, Mult. Ax.)

$\frac{81x-162}{2x-5}=81.$ (Sub. Ax.)

$\frac{x-2}{2x-5}=1.$ (Div. Ax. Dividing by 81.)

$x-2=2x-5.$ (Mult. Ax.)

$-x=-3,\ x=3.$ (Sub. and Mult. Ax.)

Let the student solve this problem by the usual method and compare the length with the solution just given.

8. $\frac{5x-3}{9}-\frac{10x+7}{18}+\frac{7x-29}{5x-12}=0.$

9. $\frac{48x+7}{24}-\frac{6x+1}{3}=\frac{x-16}{4(3x-57)}.$

10. $\frac{9(2x-3)}{14}+\frac{11x-1}{3x+1}=\frac{9x+11}{7}.$

11. $\frac{5z-6}{35}-\frac{z-5}{7z-112}-\frac{z}{7}=0.$

12. $\frac{10-x}{3}+\frac{13+x}{8}=\frac{7x-17}{x+21}-\frac{17+5x}{24}.$

137. Exercise in constructing Functions. Express the answers to the following questions in algebraic language. To test the correctness of the answers, assign some value or values to the letter or letters in both the given problem and the answer found, and see if the resulting numerical values agree. (See § 130.)

1. If a man earns \$$a$ a month and spends \$$x$ a quarter, how much will he save in a year? *Ans.* $12a-4x$ dollars.

CHECK. If he earns \$50 a month and spends \$100 a quarter, he saves \$200; also, $12a-4x=200$.

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16. If n is an integer, express an even number; an odd number; three consecutive integers; three consecutive even integers; three consecutive odd integers. *Ans.* to 1st, $2n$; to 2d, $2n+1$.

17. Find the sum of ten consecutive integers of which the first is n. Apply your result to finding the sum of 237,130, 237,131, 237,132, ⋯, to ten numbers in a short way.

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1. If to 9 times a number 4 is added, the result is 1120.

SOLUTION. $9x \qquad\qquad +4 \qquad\qquad\qquad = 1120,$

$\therefore x = 124.$

VERIFICATION. $9 \times 124 + 4 = 1120.$

2. The sum of two numbers, one 12 greater than the other, is 198.

3. Two numbers, the second of which is 3 more than twice the first, being added give 48.

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139. **Exercise in translating Equations back into Sentences.**

1. $3(x-2)=2(2x-13)$.

Ans. There is a certain number such that three times the difference between it and 2 is the same number as twice the difference between twice the number and 13.

2. $5x-4=3(x+4)$.

3. $\frac{7}{8}x-\frac{5}{6}=\frac{1}{4}x$.

4. $\frac{m-4}{4}=6-\frac{m}{3}$.

5. $\frac{7x+1}{5}-2x+\frac{4x+7}{7}=1$.

6. If B's money is $5 greater than A's, C's is $7 more than wice A's, and D's is $3 less than the sum of B's and C's, change he following equation into a sentence, making the conditions ust given a part of the sentence:

$$x+(x+5)+(2x+7)+(3x+12-3)=168.$$

7. Change $y+(y+2)+(y+4)+(y+6)=96$ into a sentence.

III. PROBLEMS CONTAINING ONE UNKNOWN QUANTITY

140. Problems involving the Solution of Equations containing One Unknown Quantity.

The solution of *problems* consists of two parts:

(1) The translating of the problem into algebraic language, called the "statement" or "stating" of the problem.

(2) The solution of the equation.

Besides the solution there is always the verification. The verification of a *problem* is accomplished, not by substituting the answer in the equation found, but through testing it by the language of the problem, since the greatest danger of mistake here lies in the stating of the problem and not in the solution of the equation.

1. What number is that whose half, third, and 100 more equals twice the number and 2 more?

2. What number is that whose half, third, and fourth parts together equal 65?

Many of the types of problems now found in algebras have come down to us from remote antiquity. Some of them were given in arithmetics, and were called puzzles or enigmas because they were quite difficult to solve by arithmetic. Some authors called them amusing, or joke problems. The purpose in giving them was to sharpen the wits, to offer recreation, and to while away the time. We give a short list of such problems.

3. A tower standing for $\frac{1}{4}$ of its length in the earth and $\frac{1}{3}$ in the water rises 100 feet in the air. What is its total height?

(From Bamberg Arithmetic, the first printed arithmetic, 1482.)

4. A vinedresser hires a laborer with the understanding that for every day he worked he should receive 10 M., and for every day he was idle he should forfeit 12 M. After 40 da. the laborer quit, receiving no money. How many days did he work?

(Bamberg Arithmetic.)

5. Heap (the unknown), its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole, it makes 33. What is the number? (Ahmes' papyrus, 1700 ? B.C.)

6. A barrel has three faucets, by the first of which it is emptied in 2 days, by the second in 3 days, and by the third in 4 days. In what time will the barrel be emptied if all are left open? (Bamberg Arithmetic.)

7. A hound pursues a hare which is 100 leaps ahead of the hound. If the hare makes 12 leaps while the hound makes 15, how many leaps will the hare make before the hound catches her, if the hound's leaps are the same length as those of the hare's? (Bamberg Arithmetic.)

8. At what time between 3 and 4 o'clock are the hands of a clock (1) together; (2) at right angles; (3) opposite.

SUGGESTION. — In solving clock problems use minute spaces as the unit of measure, and note that in any specified time whatever the minute hand traverses exactly 12 times as many spaces as the hour hand.

9. In an old Chinese arithmetic whose origin antedates 1250 B.C., there is given this problem: In the middle of a square pond 12 units on a side there grows a reed which rises 1 unit above the water. If the reed be pulled over to the middle point of one of the edges, it just reaches the top of the water. What is the depth of the water in the center?

10. A thief who stole a sum of gold was obliged to pass three doorkeepers on his way out. The first of these demanded the half of the stolen amount, but gave back to the thief 100 florins. The second demanded the half of the remainder, and then gave back 50 florins. The third demanded half of the last remaining sum, and gave back 25 florins to the thief, who now found he had left only one third of the sum stolen. How many florins did the thief steal? (Jacob Koebel, 1470–1533 A.D.)

11. A merchant adds annually $\frac{1}{3}$ of his money diminished by £100 which he spends each year, and after three years is twice as rich as at first. Find the amount of his money at first. (From Newton's Universal Arithmetic, *i.e.* Algebra, 1707.)

12. A scheme for one person to tell to another the result of a series of operations performed on a number secretly selected by the latter. Example. Denote by x the number secretly selected. Tell the person to take *any* number he wishes, and multiply it by 6, say, add 4 to the result, then add 15, then subtract 1, then divide the remainder by 3, then subtract twice the number he started with, then to square the result, subtract 6 from it, and divide the result by 5. Now tell him he has 6 for answer. Explain how you know by algebra.

13. Two parallel chords equally distant from the center of a circle intercept arcs whose sum is $\frac{2}{15}$ of the remaining parts of the circumference. Find the number of degrees in each part of the circumference.

14. Two lines intersect so that $2n - 1$ degrees represents the size of one of the angles of the two pairs of vertical angles and $3n + 6$ the number of degrees in one of the angles of the other pair. Find the number of degrees in each of the four angles.

15. If two lines are cut by a transversal so that $2n - 5$ is the number of degrees in one of the interior angles and $10n - 50$ the sum of all four interior angles, find the number of degrees in each of the eight angles formed by the transversal with the two parallel lines.

16. The perimeter of a parallelogram is 360 rd. and the ratio of two adjacent sides is 2 : 5. How many rods are there in each side?

17. Two adjacent angles of a parallelogram are $2n + 10$ and $3n - 20$ degrees. How many degrees are there in each angle?

18. An inscribed angle is subtended by $2n - 10$ degrees and the rest of the circumference is denoted by $5n + 20$ degrees. How many degrees are there in the angle?

19. The arcs intercepted between two vertical angles formed by two chords which intersect within a circle are denoted by $3y - 16$ degrees and $2y + 7$ degrees. The remaining arcs of the circumference together equal $7y - 15$ degrees. How many degrees are there in the angle between the chords?

IV. SIMULTANEOUS EQUATIONS

141. **Simultaneous Equations** are sets of equations whose unknown quantities have the same values in the different equations. Thus in $\begin{cases} 2x + 3y = 19, \\ 13x - 2y = 16, \end{cases}$ x has the value 2 and y the value 5 in both equations. Simultaneous equations are solved by a process called elimination.

142. **Elimination** is the process of deriving from two or more equations containing two or more unknown quantities a single equation containing one unknown quantity. The two most important methods of elimination are:

(1) By Substitution. (2) By Addition and Subtraction.

143. **Elimination by Substitution** is accomplished by finding the value of one unknown from one of the given equations and *substituting* this value for the same unknown in the other equation.

a. Since two equations are being handled, in order to refer to them conveniently, they are numbered (1) and (2). Changed forms of these equations will be marked (1_1), (2_1), (1_2), etc.

1. Given $\begin{cases} (1)\ \dfrac{7}{x-3} = \dfrac{8}{y-5}, \\ (2)\ \dfrac{9}{2x-1} = \dfrac{5}{3y+4}, \end{cases}$ to find values of x and y which satisfy both equations.

Solution. (1_1) $7y - 35 = 8x - 24$ (Mult. Ax.)

(1_2) $7y = 8x + 11$ (Sub. Ax.)

(1_3) $y = \dfrac{8x + 11}{7}$ (Div. Ax.)

(2_1) $27y + 36 = 10x - 5$ (Mult. Ax.)

$27\left(\dfrac{8x + 11}{7}\right) + 36 = 10x - 5$ (§ 143.)

$216x + 297 + 252 = 70x - 35$ (§ 105, Mult. Ax.)

$146x = -584$ (Sub. Ax.)

$x = -4.$ *Ans.* (Div. Ax.)

(1_3) $y = \dfrac{8 \times -4 + 11}{7} = -3.$ *Ans.* (Substituting -4 for x.)

Verification (1) $\dfrac{7}{-4-3} \equiv \dfrac{8}{-3-5}$; (2) $\dfrac{9}{-8-1} \equiv \dfrac{5}{-9+4}$.

2. $\begin{cases} (1)\ 9x + 8y = 57, \\ (2)\ 6x + 7y = 48. \end{cases}$

3. $\begin{cases} (1)\ x - \dfrac{4y-9}{11} = 5, \\ (2)\ \dfrac{9}{2} - \dfrac{x+5}{3} = -3y. \end{cases}$

144. Elimination by Addition and Subtraction.

1. Given $\begin{cases} (1)\ \dfrac{x}{4} + \dfrac{y}{5} = 5, \\ (2)\ \dfrac{2x}{3} + y = 18, \end{cases}$ to find values of x and y which satisfy both equations.

SOLUTION.

$(1_1)\ 5x + 4y = 100$ $(2_1)\ 2x + 3y = 54$ (Mult. Ax.)

$(1_2)\ 10x + 8y = 200$ (Multiplying (1_1) by 2, Mult Ax.)

$(2_2)\ 10x + 15y = 270$ (Multiplying (2_1) by 5, Mult. Ax.)

$7y = 70$ (Subtracting (1_2) from (2_2), Sub. Ax.)

$y = 10$. *Ans.* (Div. Ax.)

$(2_1)\ 2x + 3 \times 10 = 54.$ (Substituting its value for y.)

$x = 12$. *Ans.*

VERIFICATION (1) $\dfrac{12}{4} + \dfrac{10}{5} \equiv 5$; (2) $\dfrac{2 \times 12}{3} + 10 \equiv 18.$

2. **Rule.** *After reducing both equations to the form $ax + by = c$, each equation is multiplied through by such a number that the coefficients of x (or y if easier) are made the same in both new equations. Then, one of these equations is subtracted from the other (or added when the like coefficients have opposite signs).*

3. $\begin{cases} (1)\ 2x + 3y = 18, \\ (2)\ 3x - 2y = 1. \end{cases}$

4. $\begin{cases} (1)\ \dfrac{4x+5y}{40} = x - y, \\ (2)\ \dfrac{2x-y}{3} - 2y = \dfrac{1}{2}. \end{cases}$

145. Elimination by Comparison is less important than the other methods. It is performed by finding the value of the same unknown from both equations and setting these values equal to each other.

146. General Exercise in Elimination with Two Unknowns. Solve each problem by both substitution and addition and subtraction and compare the methods.

1. $\begin{cases} 2x + 7y = 38, \\ 3x + 4y = 31. \end{cases}$

2. $\begin{cases} 17a - 18b = 52, \\ 5a - 12b = 22. \end{cases}$

3. $\begin{cases} 29x + 85y = 31, \\ 13x - 43y = 95. \end{cases}$

4. $\begin{cases} .5x + y = 2.75, \\ 3.4x + .02y = 1.75. \end{cases}$

5. $\begin{cases} 5h + 2k = -7, \\ \dfrac{2h + k + 6}{h - 2k - 3} = -\dfrac{2}{7}. \end{cases}$

6. $\begin{cases} \dfrac{a + 2b}{4} + \dfrac{2a + b}{3} = 1, \\ 8a + 7b = 12. \end{cases}$

7. $\begin{cases} \dfrac{4}{m - 2} + \dfrac{3}{n + 1} = 1, \\ \dfrac{1}{m - 2} + \dfrac{6}{n + 1} = \dfrac{9}{8}. \end{cases}$

8. $\begin{cases} \dfrac{3x - 5y}{2} + 3 = \dfrac{2x + y}{5}, \\ 8 - \dfrac{x - 2y}{4} = \dfrac{x}{2} + \dfrac{y}{3}. \end{cases}$

9. $\begin{cases} \dfrac{6}{x} + \dfrac{12}{y} = -1, \\ \dfrac{8}{x} - \dfrac{9}{y} = 7. \end{cases}$

Suggestion. Solve without clearing of fractions for $\frac{1}{x}$ and $\frac{1}{y}$, since clearing introduces term xy.

147. Special Methods of Elimination. Instead of eliminating immediately, it is often better to multiply each equation by such a multiplier that when the two resulting equations are added or subtracted a new equation results, which, though still containing both unknowns, has small coefficients. This new equation can now be combined with one of the old equations, or with another new equation obtained in the same way. Any pair of such equations are said to constitute a *system*, since they suffice to find the values of the unknowns.

1. Given $\begin{cases} (1)\ 13x - 29y = 97, \\ (2)\ 17x - 39y = 129, \end{cases}$ to find values of x and y.

Solution.

$$\begin{array}{lrl} (1) & 52x - 116y = 388 & \text{(Mult. Ax.)} \\ (2) & \underline{51x - 117y = 387} & \text{(Mult. Ax.)} \\ (3) & x + \quad y = 1 & \text{(Sub. Ax.)} \\ (3_1) & 13x + \quad 13y = 13 & \text{(Ax. ?)} \\ (1) & \underline{13x - \quad 29y = 97} & \\ & 42y = -84 & \text{(Ax. ?)} \\ & y = -2. \quad \textit{Ans.} & \text{(Ax. ?)} \end{array}$$

$$(3)\ x + (-2) = 1;\ x = 3. \quad \textit{Ans.}$$

SOLUTION

$9x+3+\dfrac{81x-162}{2x-5}=18+9x+66.$ (Multiply through by 27 Mult. Ax.)

$\dfrac{81x-162}{2x-5}=81.$ (Sub. Ax.)

$\dfrac{x-2}{2x-5}=1.$ (Div. Ax. Dividing by 81.)

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4. $\frac{m-4}{4}=6-\frac{m}{3}$.

5. $\frac{7x+1}{5}-2x+\frac{4x+7}{7}=1$.

6. If B's money is $5 greater than A's, C's is $7 more than twice A's, and D's is $3 less than the sum of B's and C's, change the following equation into a sentence, making the conditions just given a part of the sentence:

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(1) The translating of the problem into algebraic language, called the "statement" or "stating" of the problem.

(2) The solution of the equation.

Besides the solution there is always the verification. The verification of a *problem* is accomplished, not by substituting the answer in the equation found, but through testing it by the language of the problem, since the greatest danger of mistake here lies in the stating of the problem and not in the solution of the equation.

1. What number is that whose half, third, and 100 more equals twice the number and 2 more?

2. What number is that whose half, third, and fourth parts together equal 65?

Many of the types of problems now found in algebras have come down to us from remote antiquity. Some of them were given in arithmetics, and were called puzzles or enigmas because they were quite difficult to solve by arithmetic. Some authors called them amusing, or joke problems. The purpose in giving them was to sharpen the wits, to offer recreation, and to while away the time. We give a short list of such problems.

3. A tower standing for $\frac{1}{4}$ of its length in the earth and $\frac{1}{3}$ in the water rises 100 feet in the air. What is its total height?

(From Bamberg Arithmetic, the first printed arithmetic, 1482.)

4. A vinedresser hires a laborer with the understanding that for every day he worked he should receive 10 M., and for every day he was idle he should forfeit 12 M. After 40 da. the laborer quit, receiving no money. How many days did he work?

(Bamberg Arithmetic.)

5. Heap (the unknown), its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole, it makes 33. What is the number? (Ahmes papyrus, 1700? B.C.)

6. A barrel has three faucets, by the first of which it is emptied in 2 days, by the second in 3 days, and by the third in 4 days. In what time will the barrel be emptied if all are left open? (Bamberg Arithmetic.)

7. A hound pursues a hare which is 100 leaps ahead of the hound. If the hare makes 12 leaps while the hound makes 15, how many leaps will the hare make before the hound catches her, if the hound's leaps are the same length as those of the hare's? (Bamberg Arithmetic.)

8. At what time between 3 and 4 o'clock are the hands of a clock (1) together; (2) at right angles; (3) opposite.

SUGGESTION. — In solving clock problems use minute spaces as the unit of measure, and note that in any specified time whatever the minute hand traverses exactly 12 times as many spaces as the hour hand.

9. In an old Chinese arithmetic whose origin antedates 1250 B.C., there is given this problem: In the middle of a square pond 12 units on a side there grows a reed which rises 1 unit above the water. If the reed be pulled over to the middle point of one of the edges, it just reaches the top of the water. What is the depth of the water in the center?

10. A thief who stole a sum of gold was obliged to pass three doorkeepers on his way out. The first of these demanded the half of the stolen amount, but gave back to the thief 100 florins. The second demanded the half of the remainder, and then gave back 50 florins. The third demanded half of the last remaining sum, and gave back 25 florins to the thief, who now found he had left only one third of the sum stolen. How many florins did the thief steal? (Jacob Koebel, 1470–1533 A.D.)

11. A merchant adds annually $\frac{1}{3}$ of his money diminished by £100 which he spends each year, and after three years is twice as rich as at first. Find the amount of his money at first. (From Newton's Universal Arithmetic, *i.e.* Algebra, 1707.)

12. A scheme for one person to tell to another the result of a series of operations performed on a number secretly selected by the latter. Example. Denote by x the number secretly selected. Tell the person to take *any* number he wishes, and multiply it by 6, say, add 4 to the result, then add 15, then subtract 1, then divide the remainder by 3, then subtract twice the number he started with, then to square the result, subtract 6 from it, and divide the result by 5. Now tell him he has 6 for answer. Explain how you know by algebra.

13. Two parallel chords equally distant from the center of a circle intercept arcs whose sum is $\frac{2}{15}$ of the remaining parts of the circumference. Find the number of degrees in each part of the circumference.

14. Two lines intersect so that $2n-1$ degrees represents the size of one of the angles of the two pairs of vertical angles and $3n+6$ the number of degrees in one of the angles of the other pair. Find the number of degrees in each of the four angles.

15. If two lines are cut by a transversal so that $2n-5$ is the number of degrees in one of the interior angles and $10n-50$ the sum of all four interior angles, find the number of degrees in each of the eight angles formed by the transversal with the two parallel lines.

16. The perimeter of a parallelogram is 360 rd. and the ratio of two adjacent sides is 2 : 5. How many rods are there in each side?

17. Two adjacent angles of a parallelogram are $2n+10$ and $3n-20$ degrees. How many degrees are there in each angle?

18. An inscribed angle is subtended by $2n-10$ degrees and the rest of the circumference is denoted by $5n+20$ degrees. How many degrees are there in the angle?

19. The arcs intercepted between two vertical angles formed by two chords which intersect within a circle are denoted by $3y-16$ degrees and $2y+7$ degrees. The remaining arcs of the circumference together equal $7y-15$ degrees. How many degrees are there in the angle between the chords?

IV. SIMULTANEOUS EQUATIONS

141. Simultaneous Equations are sets of equations whose unknown quantities have the same values in the different equations. Thus in $\begin{cases}2x+3y=19,\\13x-2y=16,\end{cases}$ x has the value 2 and y the value 5 in both equations. Simultaneous equations are solved by a process called elimination.

142. Elimination is the process of deriving from two or more equations containing two or more unknown quantities a single equation containing one unknown quantity. The two most important methods of elimination are:

(1) By Substitution. (2) By Addition and Subtraction.

143. Elimination by Substitution is accomplished by finding the value of one unknown from one of the given equations and *substituting* this value for the same unknown in the other equation.

a. Since two equations are being handled, in order to refer to them conveniently, they are numbered (1) and (2). Changed forms of these equations will be marked (1_1), (2_1), (1_2), etc.

1. Given $\begin{cases}(1)\ \dfrac{7}{x-3}=\dfrac{8}{y-5},\\[2ex](2)\ \dfrac{9}{2x-1}=\dfrac{5}{3y+4},\end{cases}$ to find values of x and y which satisfy both equations.

SOLUTION.

$$(1_1)\qquad 7y-35=8x-24 \qquad \text{(Mult. Ax.)}$$

$$(1_2)\qquad 7y=8x+11 \qquad \text{(Sub. Ax.)}$$

$$(1_3)\qquad y=\frac{8x+11}{7} \qquad \text{(Div. Ax.)}$$

$$(2_1)\qquad 27y+36=10x-5 \qquad \text{(Mult. Ax.)}$$

$$27\left(\frac{8x+11}{7}\right)+36=10x-5 \qquad \text{(§ 143.)}$$

$$216x+297+252=70x-35 \qquad \text{(§ 105, Mult. Ax.)}$$

$$146x=-584 \qquad \text{(Sub. Ax.)}$$

$$x=-4.\ \textit{Ans.} \qquad \text{(Div. Ax.)}$$

$$(1_3)\ y=\frac{8\times-4+11}{7}=-3.\ \textit{Ans.} \qquad \text{(Substituting } -4 \text{ for } x.)$$

VERIFICATION (1) $\dfrac{7}{-4-3}\equiv\dfrac{8}{-3-5}$; (2) $\dfrac{9}{-8-1}\equiv\dfrac{5}{-9+4}$.

2. $\begin{cases} (1)\ 9x + 8y = 57, \\ (2)\ 6x + 7y = 48. \end{cases}$
3. $\begin{cases} (1)\ x - \dfrac{4y-9}{11} = 5, \\ (2)\ \dfrac{9}{2} - \dfrac{x+5}{3} = -3y. \end{cases}$

144. Elimination by Addition and Subtraction.

1. Given $\begin{cases} (1)\ \dfrac{x}{4} + \dfrac{y}{5} = 5, \\ (2)\ \dfrac{2x}{3} + y = 18, \end{cases}$ to find values of x and y which satisfy both equations.

SOLUTION.

$(1_1)\quad 5x + 4y = 100 \qquad (2_1)\ 2x + 3y = 54$ (Mult. Ax.)

$(1_2)\quad 10x + 8y = 200$ (Multiplying (1_1) by 2, Mult Ax.)

$(2_2)\quad \underline{10x + 15y = 270}$ (Multiplying (2_1) by 5, Mult. Ax.)

$7y = 70$ (Subtracting (1_2) from (2_2), Sub. Ax.)

$y = 10$. *Ans.* (Div. Ax.)

$(2_1)\quad 2x + 3 \times 10 = 54.$ (Substituting its value for y.)

$x = 12$. *Ans.*

VERIFICATION (1) $\dfrac{12}{4} + \dfrac{10}{5} \equiv 5$; (2) $\dfrac{2 \times 12}{3} + 10 \equiv 18$.

2. **Rule.** *After reducing both equations to the form $ax + by = c$, each equation is multiplied through by such a number that the coefficients of x (or y if easier) are made the same in both new equations. Then, one of these equations is subtracted from the other (or added when the like coefficients have opposite signs).*

3. $\begin{cases} (1)\ 2x + 3y = 18, \\ (2)\ 3x - 2y = 1. \end{cases}$
4. $\begin{cases} (1)\ \dfrac{4x+5y}{40} = x - y, \\ (2)\ \dfrac{2x-y}{3} - 2y = \dfrac{1}{2}. \end{cases}$

145. Elimination by Comparison is less important than the other methods. It is performed by finding the value of the same unknown from both equations and setting these values equal to each other.

146. General Exercise in Elimination with Two Unknowns. Solve each problem by both substitution and addition and subtraction and compare the methods.

1. $\begin{cases} 2x + 7y = 38, \\ 3x + 4y = 31. \end{cases}$

2. $\begin{cases} 17a - 18b = 52, \\ 5a - 12b = 22. \end{cases}$

3. $\begin{cases} 29x + 85y = 31, \\ 13x - 43y = 95. \end{cases}$

4. $\begin{cases} .5x + y = 2.75, \\ 3.4x + .02y = 1.75. \end{cases}$

5. $\begin{cases} 5h + 2k = -7, \\ \dfrac{2h + k + 6}{h - 2k - 3} = -\dfrac{2}{7}. \end{cases}$

6. $\begin{cases} \dfrac{a + 2b}{4} + \dfrac{2a + b}{3} = 1, \\ 8a + 7b = 12. \end{cases}$

7. $\begin{cases} \dfrac{4}{m - 2} + \dfrac{3}{n + 1} = 1, \\ \dfrac{1}{m - 2} + \dfrac{6}{n + 1} = \dfrac{9}{8}. \end{cases}$

8. $\begin{cases} \dfrac{3x - 5y}{2} + 3 = \dfrac{2x + y}{5}, \\ 8 - \dfrac{x - 2y}{4} = \dfrac{x}{2} + \dfrac{y}{3}. \end{cases}$

9. $\begin{cases} \dfrac{6}{x} + \dfrac{12}{y} = -1, \\ \dfrac{8}{x} - \dfrac{9}{y} = 7. \end{cases}$

SUGGESTION. Solve without clearing of fractions for $\frac{1}{x}$ and $\frac{1}{y}$, since clearing introduces term xy.

147. Special Methods of Elimination. Instead of eliminating immediately, it is often better to multiply each equation by such a multiplier that when the two resulting equations are added or subtracted a new equation results, which, though still containing both unknowns, has small coefficients. This new equation can now be combined with one of the old equations, or with another new equation obtained in the same way. Any pair of such equations are said to constitute a *system*, since they suffice to find the values of the unknowns.

1. Given $\begin{cases} (1)\ 13x - 29y = 97, \\ (2)\ 17x - 39y = 129, \end{cases}$ to find values of x and y.

SOLUTION.

$$\begin{array}{lrl} (1) & 52x - 116y = 388 & \text{(Mult. Ax.)} \\ (2) & \underline{51x - 117y = 387} & \text{(Mult. Ax.)} \\ (3) & x + \quad y = 1 & \text{(Sub. Ax.)} \\ (3_1) & 13x + \ 13y = 13 & \text{(Ax. ?)} \\ (1) & \underline{13x - \ 29y = 97} & \\ & 42y = -84 & \text{(Ax. ?)} \\ & y = -2. \quad \textit{Ans.} & \text{(Ax. ?)} \end{array}$$

(3) $x + (-2) = 1$; $x = 3$. *Ans.*

Verification. (1) $13 \times 3 - 29 \times -2 \equiv 97$; (2) $17 \times 3 - 39 \times -2 \equiv 129$.

(*a*) If the student will solve this problem by the regular process of solution, that is by multiplying (1) through by 17 and (2) through by 13, he will find that the labor of solution is two or three times as great as that of the preceding solution. Evidently the plan is to multiply both members of each of the two equations by two small numbers so as to bring the corresponding coefficients *close together* in value instead of to make them equal.

2. $11x - 21y = 26$; $21x - 40y = 50$.

3. $19x + 35y = 127$; $28x + 53y = 190$.

4. $23x + 29y = -1$; $29x + 23y = 53$.

Suggestion. Add and divide through by 52, getting (3); then subtract and divide through by 6, getting (4). Use (3) and (4) to finish solution.

148. Systems of Equations. Equivalence. Two systems of equations (§ 147) are **equivalent** when one set can be derived from the other by axiomatic processes, and the values of the unknowns obtained from each set are the same. Thus, we saw in the last article that values of the unknowns derived from using a new third equation along with one of the given equations verified in both of the given equations; also in Ex. 4, that values of the unknowns obtained from the two *new* equations (3) and (4) verified in the original equations.

It may be said here that, in general, throughout simple equations, values of the unknowns found from a set of derived equations will verify in the original ones. Nevertheless the student should make it a rule to test all answers by substituting them in the original equations.

149. Simultaneous Equations Containing Three or more Unknowns.

1. Given $\begin{cases} (1)\ 2x + 3y - z = 21, \\ (2)\ 6x - 7y + 5z = 55, \\ (3)\ 9x + 5y - 2z = 71, \end{cases}$ to find values of x, y, z which will satisfy each of these equations.

Solution. Of the unknowns z is most easily eliminated.

$$\begin{array}{lrcl} (1_1) & 10x + 15y - 5z & = & 105 \quad \text{(Mult. Ax.)} \\ (2) & \underline{6x - 7y + 5z} & \underline{=} & \underline{55} \\ (4) & 16x + 8y \phantom{{}+5z} & = & 160 \quad \text{(Ax.?)} \\ (4_1) & 2x + y \phantom{{}+5z} & = & 20 \quad \text{(Ax.?)} \end{array}$$

$$
\begin{array}{lrcll}
(3) & 9x + 5y - 2z & = & 71 & \\
(1_3) & \underline{4x + 6y - 2z} & = & \underline{42} & (\text{Ax.?}) \\
(5) & 5x - y \phantom{{}-2z} & = & 29 & (\text{Ax.?}) \\
(4_1) & \underline{2x + y \phantom{{}-2z}} & = & \underline{20} & \\
 & 7x & = & 49 & (\text{Ax.?}) \\
 & x & = & 7. & \textit{Ans.}\ (\text{Ax.?}) \\
(4_1) & 2 \times 7 + y & = & 20. & (\text{Substituting its value 7 for } x). \\
 & y & = & 6. & \textit{Ans.}
\end{array}
$$

$$
\begin{array}{lrll}
(1) & 2 \times 7 + 3 \times 6 - z = 21. & & (\text{Substituting their values for } x \text{ and } y.) \\
 & z = 11. & \textit{Ans.} &
\end{array}
$$

Verification (2) $6 \times 7 - 7 \times 6 + 5 \times 11 \equiv 55$,
(3) $9 \times 7 + 5 \times 6 - 2 \times 11 \equiv 71$.

2. $\begin{cases} 4x - 3y + z = 9, \\ 9x + y - 5z = 16, \\ x - 4y + 3z = 2. \end{cases}$

3. $\begin{cases} 12l + 5m - 4n = 29, \\ 13l - 2m + 5n = 58, \\ 17l - m - n = 15. \end{cases}$

4. Make a rule for solving sets of three simultaneous equations.

5. $\begin{cases} u + v + 6t = 63, \\ v + t - 6u = -130, \\ 3u + v - t = 76. \end{cases}$

6. $\begin{cases} \frac{1}{6}x - \frac{1}{5}y + \frac{1}{4}z = 3, \\ \frac{1}{5}x - \frac{1}{4}y + \frac{1}{5}z = 1, \\ \frac{1}{4}x - \frac{1}{3}y + \frac{1}{2}z = 5. \end{cases}$

7. $\begin{cases} 2p - 3q = 3, \\ 3q - 4r = 7, \\ 4r - 5p = 2. \end{cases}$

SUGGESTION. Eliminate q, using (1) and (2), getting (4). Then take (3) and (4) together.

8. $\begin{cases} \dfrac{2}{x} + \dfrac{1}{y} - \dfrac{3}{z} = 0, \\ \dfrac{3}{z} - \dfrac{2}{y} - 2 = 0, \\ \dfrac{1}{x} + \dfrac{1}{z} - \dfrac{4}{3} = 0. \end{cases}$

SUGGESTION. Do not clear of the denominators, but solve directly for $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$, and then get x, y, z. Solve as in Ex. 7.

9. $\begin{cases} 3x + 6y + 2z + u = 2, \\ x - y - 3z - 4u = 3, \\ x + 2y - 2z - 2u = 0, \\ 2x + y - z - 3u = 5. \end{cases}$

SUGGESTION. Eliminate x first. Combine (2) and (3), eliminating x, getting (5). Then combine (1) and (2), eliminating x, getting (6). Then combine (2) and (4), eliminating x, getting (7). Now solve (5), (6), and (7) as in preceding problems.

150. Problems involving the Solution of Equations containing One or More Unknowns. The choice of the number of unknowns is left to the student. The same problem can often be solved equally well by using different numbers of unknowns.

1. A gentleman divides 2 dollars among 12 children, giving to some 18¢ each, and to the others 14¢ each. How many were of each class?

2. A grocer has two sorts of sugar, one worth 5¢ and the other $6\frac{1}{2}$¢ a pound. How many pounds of each sort must be taken to make a mixture of 100 lb. worth $5.60?

3. Two sides of an equiangular triangle are $15x-6y$ and $20x-9y$, and the third side is 15. Find x and y.

4. How many gallons each of cream containing 30% fat and milk containing 5% fat shall be mixed so as to produce 10 gal. of mixture containing 25% of fat?

5. If 6, 8, and 11 ft. respectively are the sides of a triangle, find the segments into which the bisector of the angle between 6 and 8 sides divides the third side. What are the segments when an exterior angle is bisected? (See § 51.)

6. In an athletic meet the team from town A won the championship with a total of 36 points, taking five firsts, three seconds, and two thirds. The team from town B got four firsts, two seconds, and four thirds, with a total of 30 points, and the team from town C got two firsts, six seconds, and five thirds, and a total of 33 points. How many points did a first, second, and third respectively count?

7. Earth and Venus move around the sun in orbits approximately circular, one moving around the sun in a shorter period than the other so that at one time they and the sun are in line with Venus between the other two, and at another time they are in line in a different place with the sun between the other two. When nearest together Earth and Venus are only 25.6 millions of miles from each other; when farthest apart they are 160 millions of miles distant from each other. Find the distance of each planet from the sun.

V. GRAPHICAL SOLUTION OF SIMULTANEOUS EQUATIONS

151. Value of the Graphical Method. Whether the algebraic or he graphical method will give the answer more quickly depends n the problem and the skill of the user, but the algebraic soluion is always accurate, while the graphic solution is often only pproximate. One of these methods can be used to check the nswer obtained by the other, though verification is also available or both. But for the student now the main value of the graphcal solutions consists in the fact that they throw a strong light n the nature of simultaneous equations and on peculiarities that nay occur in them.

152. Graphical Solution of Pairs of Equations containing Two nknowns. 1. Solve (1) $\frac{x}{3}-\frac{y}{6}=\frac{1}{2}$, (2) $\frac{x}{5}-\frac{3y}{10}=\frac{1}{2}$, by the raphical method, and check with an algebraic solution.

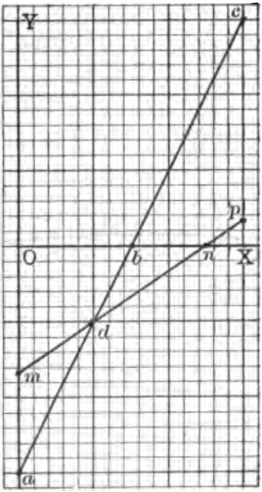
Graph Solution.

$(1_1)\quad 2x - y = 3$

(x, y)	Pt.
$(0, -3)$	a
$(1.5, 0)$	b
$(3, 3)$	c

$(2_1)\quad 2x - 3y = 5$

(x, y)	Pt.
$(0 - 1.67)$	m
$(2.5, 0)$	n
$(3, .33)$	p



The coördinates that satisfy *both* equations are those of the point d, *where the graphs cross.* The coördinates of d are $x = 1$, $y = -1$. *Answers.*

Algebraic Solution.

$$\begin{array}{rrcr}
(1_1) & 2x-y & = & 3\\
(2_1) & 2x-3y & = & 5\\
\hline
 & 2y & = & -2\\
 & y & = & -1. \quad \textit{Ans.}
\end{array}
\qquad
\begin{array}{l}
(1_1)\ 2x-(-1)=3. \ (\S\ 144.\\
\qquad\qquad x=1. \quad \textit{Ans.}
\end{array}$$

2. $\begin{cases}3x+4y=10,\\ 4x+y=9.\end{cases}$

3. $\begin{cases}5x+6y=17,\\ 6x+5y=16.\end{cases}$

4. $\begin{cases}2x+3y=43,\\ 10x-y=7.\end{cases}$

5. $\begin{cases}11x-14y=14,\\ 5x+7y=41.\end{cases}$

6. $\begin{cases}18x-10y=29,\\ 14x-15y=24.\end{cases}$

7. $\begin{cases}7x-9y=-22,\\ x=-4.\end{cases}$

8. $\begin{cases}11d+16t=64,\\ 7d-12t=13.\end{cases}$

9. $\begin{cases}8x=5y,\\ 13x=8y+1.\end{cases}$

10. $\begin{cases}11x-3y=321,\\ 5x-16y=585.\end{cases}$

11. $\begin{cases}7x+2y=76,\\ 2x-3y=11.\end{cases}$

Suggestion to Ex. 10. Since the numbers are large, the centimeter can not be used as the unit of measure. The 2-millimeter unit can be employed in this case.

153. Consistent Equations. The student may ask himself the question whether if three equations, each containing two unknowns, were taken at random, the values of x and y obtained by solving the system formed out of the first two equations would have the same values as those obtained by taking the system formed from the second and third equations, or by taking the system formed from the first and third equations.

On the other hand, if the third equation was obtained by combining the first two in some way, would the values of x and y found by solving the system formed from the first two satisfy the third equation?

Example.

$$\begin{array}{lrcr}
(1) & 11x-10y & = & 14\\
(2_1) & 10x+14y & = & 82\\
\hline
(3) & x-24y & = & -68
\end{array}
\qquad\qquad (2)\ 5x+7y=41$$

Construct now the graphs of equations (1) and (2) and (3) to the same axes. What do you learn?

Test problems of § 147 in the same way.

154. Inconsistent Equations. Construct the graphs for the following equations on the same axes:

(1) $4x-2y=7$. (2) $4x-9y=-21$. (3) $3x-4y=-8$.

Solve the system formed from (1) and (2) algebraically and compare the answers with those on diagram obtained graphically. Do the same with the system (1) and (3), and with the system (2) and (3).

Write sets of three equations at random and see whether their graphs intersect in three points or one point.

155. Systems of Equations in which One of the Two Given Equations can be derived from the Other.

Construct the graphs for the system

$$(1)\ 2x-5y=11;\quad (2)\ \frac{8x-44}{5}=4y,$$

in which (2) can be derived from (1) by first multiplying through by 4, then transposing, and then dividing through by 5.

What do you learn about these graphs?

Test your answer by making and testing other similarly constructed systems. Is then the system of marking equations, in which the equation marked (1) at the start is marked (1) with subscripts throughout, justified?

156. Systems of Equations which differ only in their Known Terms.

Construct the graphs on the same axes for the system

$$(1)\ 2x-5y=12,\quad (2)\ 4x=10y+6,$$

in which, obviously, (2) can be changed into $2x-5y=3$.

What do you find true of these graphs?

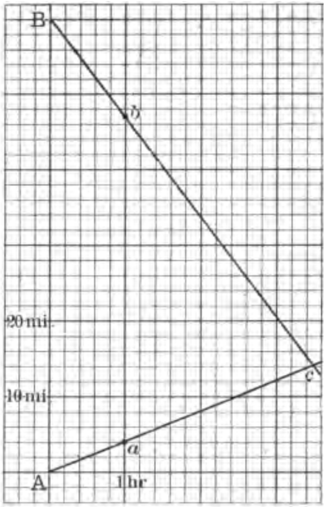
Test your answer by constructing similar systems which differ only in their known terms.

We learn then that whenever by transposing, and multiplying or dividing through, two equations can be made to differ in their known terms only, *their graphs are parallel.*

a. The graphs of three equations each having two unknowns may intersect in 0, or 1, or 2, or 3 points (not at infinity), or in every point. Explain,

157. Solution of Problems by Graphs. — Verify the results in Exs. 1–5, 8, 9, following, with algebraic solutions.

1. Two men set out from two towns A and B, 60 mi. apart, the one from A walking 4 mi. an hour, and the one from B riding in an automobile at the rate of 13 mi. an hour. How far from A will they meet, and in how many hours from the time they start?



Solution. Take a vertical axis 6 cm. long to denote 60 mi.; marking its extremities A and B, A below and B above. To the right of A let the centimeters denote hours. Then thinking of A as origin, locate point a as (1, 4) in which 1 is one hour and 4 is 4 of the 60 mi. in the line AB. Again, thinking of B as origin, locate point b as 1 hr. to the right of B and 13 of the 60 mi. units below B. Now draw Aa and Bb and extend them until they meet at c. The number of units from c down to the horizontal line through A gives the number of miles from A to where they meet. Also the abscissa of c gives the time when they meet.

Notice that $y = 4x$ is the equation of Aa, and $y = -13x$ that of Bb as in § 122, but different unit lengths are used here for x and y.

Verification by an Algebraical Solution. Let x = number of miles from A to where they meet. Then $\frac{x}{4} = \frac{(60 - x)}{13}$, whence $x = 14\frac{2}{17}$ mi. This shows that the squared paper solution is not quite accurate, but gives a good approximation to the answer.

2. Two trains start from New York and Washington at the same time, 8 A.M., the one from New York running 45 mi. an hour and the other 40 mi. an hour, and Washington and New York are 228 mi. apart. How far from New York will they meet, and at what hour of the day?

3. Two steamers leave New York and Queenstown, Ireland, the same day, Monday, the first at 3 P.M. and the other at 6 A.M., both Greenwich time. The first makes 600 and the second 500 geographical miles a day, and the two places are 2780 geographical miles apart. How far from Queenstown will they meet, and at what time?

SUGGESTION. Let the vertical line AB denote 2780 mi., A being New York. Then start the graph for the New York steamer 9 hr. to the right of A as origin, point A being at 6 A.M.

4. A man made a trip to the country with an automobile, running 18 mi. an hour. The machine broke down, and he had to return on a bicycle, making 10 mi. an hour. He was away from home 3½ hours in all; how far did he go?

SUGGESTION. Let MN represent a horizontal line which stands for 3½ hr. Draw the graphs for the motion from the points M and N.

5. The 20th Century Limited leaves New York (1911 schedule) at 4 P.M., traveling 53 mi. an hour, following the Day Express which leaves New York at 9.40 A.M., traveling 34 mi. an hour. How far from New York and at what time will the fast train catch the other?

SUGGESTION. Let MN represent the difference in time of starting. Draw both graphs northeast from M and N.

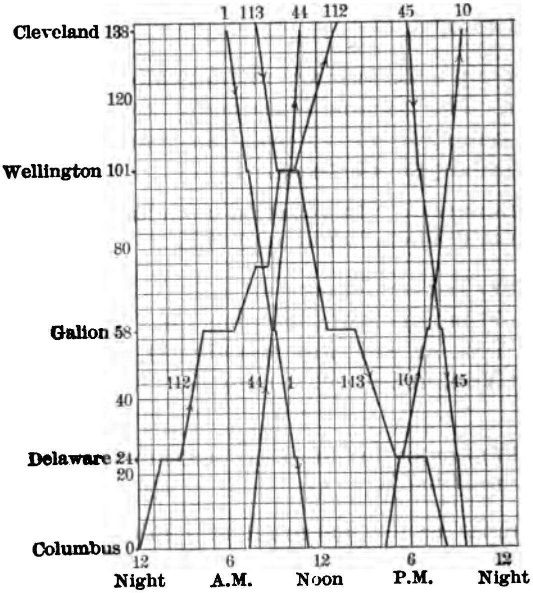
6. Construct a graph for the movement of a suburban train which runs on the following schedule:

STA.	MI.	ARRIVE	DEPART
A	0		6 A.M.
B	7	6.15	6.18
C	12	6.25	6.27
D	18	6.35	6.40
E	25	6.50	

SUGGESTION. Notice when the train is standing still the graph runs horizontally, so that the graph for a train stopping and starting is zigzag.

7. We give next a "*train chart*" used by train dispatchers on railroads to make time schedules. On a large chart lines are drawn (or threads are stretched, held in position by pins), as in

accompanying diagram, which show where and when trains start, stop, and pass each other. The stops of the four passenger trains (Nos. 1, 10, 44, 45) are so short that they hardly show. The freight (Nos. 112, 113) stops are much longer. Any number of trains can be put on the same diagram.



8. A can do a job in 10 days and B the same in 13 days. In what time can both do it?

SUGGESTION. Let any convenient vertical length AB denote the work, *i.e.* one times the work. Mark the days to the right from both A and B. Let a be 10 days directly east from B and b 13 days east of A. Join A and a and B and b. Where these graphs cross is the answer sought.

9. Three faucets fill a cistern, the first in 20 min., the second in 30 min., and the third in 40 min. In what time will all fill it?

SUGGESTION. First find the time for the first two, and then for this result and the third.

VI. LITERAL EQUATIONS

158. Solution of Applied Formulas for any Letter. The following formulas come from arithmetic, geometry, physics (including sound, light, heat, electricity), shop-work, trigonometry, engineering, etc. They are all solved by use of the axioms in much the same manner as the equations of § 135.

Solve the following formulas for each letter in them (excepting known numbers like π) not given directly:

1. $s = ba$ (area of a rectangle = base × altitude.)

Solution. $ba = s$. (Reversing members.)

$a = \frac{s}{b}$. *Ans.* (Div. Ax.). $b = \frac{s}{a}$. *Ans.* (Div. Ax.)

2. $c = \frac{b}{h}$.
3. $s = \frac{a}{h}$.
4. $t = \frac{a}{b}$.
5. $d = \frac{w}{v}$.
6. $w = fd$.
7. $c = pn$.
8. $a = \frac{1}{2}bh$.
9. $A = \pi ab$.
10. $ST = st$.
11. $t = \frac{1}{k}$.
12. $F = \frac{Wa}{g}$.
13. $l = \frac{\pi dn}{360}$.
14. $\frac{PV}{T} = \frac{P'V'}{T'}$.
15. $E = \frac{nCN}{10^8}$.
16. $R = 3 + \frac{V}{6}$.
17. $P \times P_a \times S_a = W \times W_a \times s_a$.
18. $b = 1.155(f - 0.2)$
19. $R = 0.3788\,GT$.
20. $\frac{P}{W} = \frac{dr}{2\pi lR}$.
21. $a = b(1 + r)$.
22. $a = p(1 + rt)$.
23. $v = 4N(l + r)$.
24. $Mts = M't's'$.
25. $d = d' + \frac{2}{p}$.
26. $A = \frac{1}{2}a(b + b')$.
27. $H = \frac{PLAN}{33000}$.
28. $C = \frac{SE}{Sb + R}$.

Solution for S. $SCb + CR = SE$. (Mult. Ax.)

$SCb - SE = -CR$. (Sub. Ax. Unknowns to left, knowns to right member.)

$S(E - Cb) = CR$. (Changing signs, Mult. Ax., and factoring.)

$S = \frac{CR}{E - Cb}$. *Ans.* (Div. Ax.)

Other answers:

$$E = \frac{SbC + RC}{S};\; R = \frac{SE - SbC}{C};\; b = \frac{SE - CR}{CS}.$$

29. $C=\frac{EP}{b+RP}.$ 30. $c=\frac{SEP}{Sb+RP}.$ 31. $R=\frac{gs}{g+s}.$

32. $s=\frac{lr-a}{r-1}.$ 33. $S=\frac{W}{W-W'}.$ 34. $\frac{P}{W}=\frac{p}{2\pi R}.$

35. $\frac{1}{F}=\frac{1}{D}+\frac{1}{D'}.$ 36. $S=\frac{w_1}{w_1+w_2-w_3}.$ 37. $\frac{2R-h}{h}=\frac{l}{2p}.$

38. $\frac{P+p}{2p}=\frac{P}{P'}.$ 39. $l=\frac{\pi D+\pi D'}{2}+2d.$ 40. $a=\frac{2n-4}{n}.$

41. $\frac{1}{R}=\frac{1}{r}+\frac{1}{r'}+\frac{1}{r''}.$ 42. $ms(t-t_m)=m's'(t_m-t').$

Pupils should examine geometries, books on physics, and shop mathematics for similar formulas, and solve those found for any letter. But if in any formula the unknown appears to the second or higher powers, the method of solution is different from that here explained. It will be given later.

159. Making and Using of Formulas in Arithmetic.

1. If b represents base, r rate, and p percentage, make a formula giving the value of p. Solve this formula for b; for r.

a. The word rate means *ratio*, that is, the ratio of the percentage to the base; or, if r = rate, $r=\frac{p}{b}$. In the solution of problems it is often preferable to let $r\%$, or $\frac{r}{100}$ = the rate, thus making r ordinarily an integral or mixed number, rather than a decimal. Then $\frac{r}{100}=\frac{p}{b}.$

Solve the following numerical problems by substituting in the appropriate formula:

2. A trader having $1960 spent 15 % of it. What sum did he spend?

3. If I sell $\frac{5}{9}$ of a carload of wheat for what $\frac{2}{3}$ of the whole carload cost, what is the gain per cent?

Suggestion. Find first what part of cost whole carload would be sold for and then find gain.

4. $68\frac{3}{4}$ is $9\frac{1}{6}\%$ of what number?

5. If b represents base, r rate, a amount, or sum of base and percentage, and d "difference," or base less percentage, write a formula for value of a; also a formula for value of d. Solve the latter for b.

6. A has \$3009, which is 18 % more than B has. How much has B? Also A has 15 % less than C. How much has C?

7. If p represents principal, r rate, t number of years principal bore simple interest, and a amount, make a formula giving the value of a.

a. This formula can be used to solve *any* problem in simple interest.

8. Find the amount of \$520 at 5 % for 4 yr. 4 mo. 24 da. by using formula.

9. Find the rate when \$1652.64 is amount, principal is \$1320, and time is 3 yr. 7 mo. 6 da. Solve the formula for r first.

10. Find the principal when the amount is \$279.18, rate is 3 %, and time is 1 yr. 1 mo. 6 da. Solve the formula for p first.

11. Find the time when the amount is \$1148, the rate is $4\frac{1}{2}\%$, and the principal is \$1025. Solve the formula for t first.

12. Potatoes whose value was p dollars were shipped to a commission merchant to sell, who charged $r\%$ for selling or buying. With proceeds from the sale of the potatoes he was instructed to buy salt. What is the value of the salt he should ship back? Suppose $p = 3198$ and $r = 2\frac{1}{2}$.

13. If goods that cost \$$c$ are sold for \$$s$, what is the rate per cent r of gain? Find rate r when $c = 450$, $s = 531$.

14. Goods that cost c¢ were marked to sell at m¢, but $r'\%$ was thrown off from the marked price. What was the actual gain per cent r?

Given $c = 8$, $m = 12$, $r' = 16\frac{2}{3}$, to find r.

15. Three discounts of $r\%$, $s\%$, $t\%$ were taken from a bill of \$$b$. What was the net cost c?

Given $b = 2250$, $r = 20$, $s = 15$, $t = 8$, to find c.

16. Find the amount $\$a$ of a note of $\$p$ which drew simple interest at $r\%$ for n yr. m mo. d da., assuming 12 mo. of 30 da. each to a month. Then find a when $p=400$, $r=6$, $n=3$, $m=9$, $d=13$.

17. Find B's tax $\$t$ on $\$p$ worth of property when the total assessed valuation is $\$A$ and the tax to be raised is T. Find t when $p=4000$, $A=1{,}946{,}500$, and $T=33090.50$.

18. Find the insurance premium $\$p$ to be paid on a house valued at $\$h$ when insured for $v\%$ of its valuation at an $r\%$ rate. A carriage factory and stock worth $40,000 were insured for 90% of their value at $3\frac{1}{2}\%$. Find p.

19. What is the bank discount $\$d$ on a non-interest-bearing note for $\$a$ if discounted at $r\%$, n days before it is due? Find discount $\$d$ on a note for $300 discounted at 7% 45 days before it was due.

20. A railroad stock is quoted at $n\%$ above par and its annual dividends are $r\%$; what interest rate R is realized by an investor? Suppose $n=28$, $r=8$, to find R.

21. What is the duty on n yd. of cloth worth $\$c$ a yard, taxed at $r\%$ ad valorem, plus a specific duty of s¢ a yard? What would be the duty on 75 yd. of cloth worth $2 a yard, at 55% ad valorem, and 11¢ a yard?

22. The interest on $\$x$ for t yr. and m mo. at a certain rate per cent was $\$i$. What was the rate? Find rate when $t=2$, $m=3$, $i=69.75$, and $x=620$.

23. The population of a town in 1900 was p and in 1910 it was q. What was the gain per cent r? Make the calculation for the following cities:

	p	q
Cleveland, O.,	381,768	560,663
Atlanta, Ga.,	89,872	154,839
Seattle, Wash.,	80,671	237,194
Los Angeles,	102,479	319,198
New York,	3,437,202	4,766,883

24. A draft on New York was bought at $d\%$ discount for $\$k$. What was its face $\$f$? Find f when $k=480$ and $d=\frac{1}{2}$.

25. When a stock is quoted at $h\%$ below par and pays $d\%$ annual dividends, how much money $\$x$ must be invested to yield the purchaser an income of $\$i$ annually?

Given $h=8$, $d=5$, $i=4000$.

160. Solution of Literal Equations containing Two Unknown Quantities.

1. Given (1) $ax+by=c$; (2) $a'x+b'y=c'$.

SOLUTION

$$\begin{array}{llllr}
(2_1) & aa'x+ab'y=ac' & (1_2) & ab'x+bb'y=b'c & (\text{Ax. ?}) \\
(1_1) & \underline{aa'x+a'by=a'c} & (2_2) & \underline{a'bx+bb'y=bc'} & (\text{Ax. ?}) \\
 & ab'y-a'by=ac'-a'c & & ab'x-a'bx=b'c-bc' & (\text{Ax. ?})
\end{array}$$

$$y=\frac{ac'-a'c}{ab'-a'b}.\quad \textit{Ans.} \qquad\qquad x=\frac{b'c-bc'}{ab'-a'b}.\quad \textit{Ans.}$$

VERIFICATION (1) $\quad a\cdot\dfrac{b'c-bc'}{ab'-a'b}+b\cdot\dfrac{ac'-a'c}{ab'-a'b}=c,$

or, $\quad ab'c-abc'+abc'-a'bc\equiv ab'c-a'bc.$ (Mult. Ax.)

2. (1) $ax-by=0$; (2) $x+y=c$.

3. (1) $x+ay=b$; (2) $ax-by=c$.

4. (1) $3ax-2by=c$; (2) $a^2x+b^2y=5bc$.

5. (1) $p=a+(m-1)d$; (2) $q=a+(n-1)d$; a and d being regarded as the unknowns.

6. (1) $\dfrac{x}{b+c}+\dfrac{y}{a+c}=2$; (2) $ax-by=ac-bc$.

7. (1) $\dfrac{u}{a+b}+\dfrac{v}{a-b}=2a$; (2) $u-v=4ab$.

161. Literal Equations containing more than Two Unknowns.

1. $\begin{cases}(1)\ ax+by=c,\\ (2)\ cx+az=b,\\ (3)\ bz+cy=a.\end{cases}$ (§ 149, 7.) $\qquad$ **2.** $\begin{cases}ax+by+cz=a,\\ ax-by-cz=b,\\ ax+cy+bz=c.\end{cases}$

162. Problems involving the Solution of Literal Equations.

1. A merchant has two kinds of sugar that he sells at a¢ and b¢ respectively. He desires to make a mixture of c lb. that will sell for d¢ a pound. How many pounds of each kind shall he take?

2. A man distributed a¢ among n persons, some receiving b¢ each, and others c¢ each. How many received b¢ each, and how many c¢?

3. A was m times as old as B a years ago, and will be n times as old as B in b years. Find the age of each at present.

4. A and B can do a piece of work in a days, or if A works m days alone, B can finish the work by working n days. In how many days can each do the work? (See § 146, 9.)

5. A man invested p dollars, part at r%, and the remainder at s%. His annual income from both investments was i dollars. What was the amount of each investment?

6. Two trains are scheduled to leave A and B, d miles apart, at the same time and to meet in h hours. If the train that leaves B is b hours late, and runs at its usual rate, it will meet the other train in k hours. What is the rate of each train?

7. The average age of A, B, and C is m years. The average age of A and B is n years, and of B and C is p years. What are their ages?

CHAPTER VI

INVOLUTION AND EVOLUTION

I. INVOLUTION

163. Involution is the operation of raising quantities to powers.

164. Law for Raising Quantities to Powers. Let it be required to show that $(x^m)^n = x^{mn}$, m and n being integers.

$(x^m)^n = (xxx \cdots$ to m factors$) \times (xxx \cdots$ to m factors$) \times \cdots$ to n times.

$= xxxx \cdots$ to mn times.

$= x^{mn}$.

Hence, to raise a quantity with any integral exponent to any integral power, multiply the exponent of the quantity by the index of the power for the exponent of the quantity in the result.

165. Raising Monomials to Powers.

Type form: $(x^a y^b)^n = x^{an} y^{bn}$.

1. Square: $3m^2$; $-5a$; $-6a^m$; $\frac{2}{3}x^3$; $-3a^3b^n$.

2. Cube: $-6xy$; $3abc$; $-2m^2y$; $\frac{2}{5}a^2bc$; $-3a^mb^n$.

3. Simplify: $(3ab)^4$; $(2 \times 3^2 \times 5)^3$; $(3 \times 2a)^4$; $(-\frac{2}{5}bc^n)^3$; $(2^2 \times 5)^4$; $(a^m)^{2p}$; $(a^m b^p c^q)^3$; $(2^2 \times 3^3 \times 5)^n$; $(4^2 \times 5^n \times 7)^a$.

4. Simplify: $3(2x)^2$ (see § 21); $5(2 \times 3a)^4(3b)^2$; $\frac{1}{8}(\frac{1}{2}a)^2(4b)^3$; $6(-3b)^2(-2c)^3$; $2(3 \times 4)^2(2 \times 5)^5$; $2(2a)^n(2b)^m$.

166. Raising Binomials to Powers. Newton's Theorem. By a theorem, or by actual multiplication, show that:

$$(a-b)^2 = a^2 - 2ab + b^2.$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$
$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$
$$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

Examining the right members of these equations, we see;

(1) *That the signs are alternately* + *and* −.

(2) *That the exponent of* a *in the first term in each case is the same as the exponent of* $(a - b)$ *in the left member, and that the exponents of* a *decrease by 1 from term to term; also that* b *appears first in each case in the second term, and its exponents increase by 1 from term to term.*

(3) *That the first coefficient in the right member in each case is 1 (understood), and the second coefficient is the same as the exponent of* $(a - b)$ *in the left member; also that each succeeding coefficient can be obtained by multiplying the coefficient of the preceding term by the exponent of the leading letter* a, *and dividing the product by the exponent of the other letter increased by 1.*

1. Raise $(a - b)$ to the 8th power.

Solution. $(a - b)^8 = a^8 - 8\,a^7b + 28\,a^6b^2 - 56\,a^5b^3 + 70\,a^4b^4$
$- 56\,a^3b^5 + 28\,a^2b^6 - 8\,ab^7 + b^8.$

2. $(a - b)^{10}$. **3.** $(m + n)^6$. **4.** $(H - K)^9$.

5. Raise $2\,a^2 - 3\,b$ to the fifth power.

Solution. $(x - y)^5 = x^5 - 5x^4y + 10\,x^3y^2 - 10\,x^2y^3 + 5\,xy^4 - y^5.$

Then, $(2\,a^2 - 3\,b)^5 = (2\,a^2)^5 - 5(2\,a^2)^4(3\,b) + 10(2\,a^2)^3(3\,b)^2$
$- 10(2\,a^2)^2(3\,b)^3 + 5(2\,a^2)(3\,b)^4 - (3\,b)^5$

(See § 21, 1.) $= 32\,a^{10} - 240\,a^8b + 720\,a^6b^2 - 1080\,a^4b^3 + 810\,a^2b^4$
$- 243\,b^5.$

6. $(3\,m^2 - 5\,n)^3$. **7.** $(2\,m^2n^3 - 1)^6$. **8.** $(4\,m^n - \frac{1}{3}p^3)^4$.

9. $(10 - 1)^4$. **10.** $(14\frac{1}{4})^3 = (14 + \frac{1}{4})^3$. **11.** $(12\frac{1}{6})^3$

In cases of raising a sum or difference of two quantities of which one is small as compared with the other, Newton's theorem furnishes a quickly and easily obtained *approximate* result, since after the first or second terms, the numerical values of the succeeding terms diminish rapidly and some or all of the latter terms can be neglected. Thus,

$$(122\tfrac{1}{3})^3 = 122^3 + \cancel{3} \times 122^2 \times \frac{1}{\cancel{3}} + 3 \times 122 \times \frac{1}{9} + \frac{1}{27}$$
$$= 122^2(122 + 1) + 41 \text{ nearly, or } 1{,}830{,}773.$$

12. $(100-\frac{1}{2})^3$. **13.** $(18\frac{1}{6})^3$. **14.** $(2\frac{1}{16})^4$.

15. $(5\frac{19}{20})^5=(6-\frac{1}{20})^5$. **16.** $(1\frac{11}{12})^6$. **17.** 999^4.

167. Raising Polynomials to Powers. This is accomplished by changing the given polynomial into a binomial by the use of parenthesis, and applying Newton's theorem.

1. Cube $2a+3b-c$.

SOLUTION. $(2a+3b-c)^3=[(2a+3b)-c]^3$

$$=(2a+3b)^3-3(2a+3b)^2c+3(2a+3b)c^2-c^3.$$

The solution is continued by expanding each term of this result, then multiplying and combining terms when this can be done.

2. $(m^2-3mn+4n^2)^3$. **3.** $(a^m+a^n-2c^{2p})^3$.

4. *Indicate* the expansion of $(2x-3y+4z-5n)^4$ without expanding the several terms.

II. EVOLUTION

168. Evolution is the process of extracting roots, that is, of finding equal factors of quantities.

169. Law for Extracting Roots.

Let it be required to show that $\sqrt[q]{x^{pq}}=x^p$.

We have, $x^{pq}=(x^p)^q$. (By § 164.)

Then, $\sqrt[q]{(x^p)^q}=x^p$. (Since the qth root of the qth power of a quantity equals that quantity, by the definitions of power and root, §§ 12, 13.)

Hence, to extract a root of a quantity, divide the exponent of the quantity by the index (§ 13) *of the root for the exponent of the quantity in the answer.*

170. Extraction of the Roots of Monomials.

Type form: $\sqrt[n]{a^{mn}b^{np}}=a^mb^p$.

Even roots of positive quantities can have for sign either + or − and are marked ± (see § 44, 17). Even roots of negative quantities cannot be real quantities. Thus, $\sqrt{-4x^2}$ is not $2x$, since $(2x)^2=+4x^2$; neither is it $-2x$, since $(-2x)^2$ also equals $+4x^2$, and not $-4x^2$. Even roots of negative quantities have been called **imaginaries**,

1. $\sqrt{36\,a^2}$.
2. $\sqrt[3]{-64\,a^3b^6}$.
3. $\sqrt[5]{32\,a^{10}}$.
4. $\sqrt[m]{a^{2m}b^{5m}c^{3m}}$.
5. $\sqrt{8 \times 3^3 \times 6}$.
6. $\sqrt[3]{-2^6 \times 5^3 \times 7^9}$.
7. $\sqrt[4]{\dfrac{16\,a^4}{81\,b^8}}$.
8. $\sqrt[10]{\dfrac{x^{10}y^{20}}{z^{30}}}$.
9. $\sqrt{\dfrac{5 \times 3^2 \times 2^8}{2^2 \times 5^{13} \times 3^5}}$.

171. Square Root of Polynomials.

Let it be required to derive a process for the extraction of the square root of a polynomial from a study of the formula

$$a^2 + 2\,ab + b^2 = (a + b)^2.$$

SOLUTION.

$$\begin{array}{r|l}
 & a^2 + 2\,ab + b^2 \;\underline{|\; a + b} \\
 & a^2 \\
\hline
2\,a + b & 2\,ab + b^2 \\
 & \underline{2\,ab + b^2}
\end{array}$$

How is a, the first term of the root, found? How is $2\,a$, the trial divisor, found from the first term of the root? How is b, the second term of the root, found? In what two places is b written when found? How is the operation concluded?

1. Extract square root of $4\,x^4 - 12\,x^3 + 5\,x^2 + 6\,x + 1$.

SOLUTION.

$$\begin{array}{r|l}
 & 4\,x^4 - 12\,x^3 + 5\,x^2 + 6\,x + 1 \;\underline{|\; 2\,x^2 - 3\,x - 1 \;\; \textit{Ans.}} \\
 & 4\,x^4 \\
\hline
4\,x^2 - 3\,x & -12\,x^3 + 5\,x^2 \\
 & \underline{-12\,x^3 + 9\,x^2} \\
(4\,x^2 - 6\,x) - 1 & -4\,x^2 + 6\,x + 1 \\
 & \underline{-4\,x^2 + 6\,x + 1}
\end{array}$$

REMARK. After two terms of the root are found, to find the next trial divisor, the two terms are regarded as one quantity or as a monomial and are doubled, giving $4\,x^2 - 6\,x$.

2. $81\,x^4 - 432\,x^3 + 864\,x^2 - 768\,x + 256$.

3. $1 - 2\,z + 2\,z^2 - z^3 + \dfrac{z^4}{4}$.

4. Extract fourth root, *i.e.* square root twice, of $a^4 + 8\,a^3b + 24\,a^2b^2 + 32\,ab^3 + 16\,b^4$.

5. Make a rule for the extraction of the square root of algebraical quantities, mentioning first the matter of arrangement of terms before beginning.

6. $x^6 + 25x^2 + 10x^4 - 4x^5 - 20x^3 + 16 - 24x.$

7. $\sqrt{x(x+1)(x+2)(x+3)+1} = ?$

Solve the two following mentally by getting the first and last terms, and then the middle one. Test the answer carefully, paying special attention to the sign of the last term.

8. $\sqrt{x^4 - 2x^3 + 3x^2 - 2x + 1} = ?$

9. $\sqrt{9x^4 - 12x^3 + 16x^2 - 8x + 4} = ?$

172. Extraction of the Square Root of Arithmetical Numbers.

1. Extract the square root of 1156.

Solution. Since 1156 lies between 900 and 1600, its square root lies between 30 and 40, *i.e.* between 3 tens and 4 tens.

In general counting from the unit's order, each pair of figures, or "period," in the number gives one figure in the root, except that the left-hand period may have only one figure. Thus, the square root of 1'49 is 12+. The periods are usually marked by little lines above and between the figures.

For guidance in the solution the following formula is used:

$$(t+u)^2 = t^2 + 2tu + u^2 = t^2 + (2t+u)u.$$

Notice in the last expression that $2t$ is the *trial* divisor, and $2t+u$ is the *complete* divisor. The process at the right below is just like that at the left. That at the left was explained in the last article.

$$\begin{array}{r|l}
 & t^2 + 2tu + u^2 \;\big|\; \underline{t+u} \\
 & \underline{t^2 \qquad\qquad} \\
2t+u & 2tu + u^2 \\
 & \underline{2tu + u^2}
\end{array}$$

$$\begin{array}{r|l}
 & \qquad\quad t + u \\
 & 11'56 \;\big|\; \underline{30+4} \\
t^2 = & \underline{9\,00} \\
\text{Trial divisor} = 2t = 60 & 2\,56 \\
u = \;\;4 & \\
\text{Complete divisor} = 2t + u = 64 & \underline{2\,56}
\end{array}$$

2. Extract the square root of 43,347.24 and prove answer.

Solution.

$$\begin{array}{r|l}
 & 4'33'47'.24 \;\big|\; \underline{208.2} \\
 & \underline{4 \qquad\quad} \\
408 & 33\,47 \\
 & \underline{32\,64} \\
4162 & 83\,24 \\
 & \underline{83\,24}
\end{array}$$

$(208.2)^2 = 43347.24$ by actual multiplication.

Explanation. In the solution to Ex. 1, the ciphers were retained to make the process clear. In this solution no cipher is written that is not essential to the solution.

When the first figure of the root is doubled for trial divisor it is *understood* to be 40, though no zero is written after 4 at first. Now 40 is not contained in the period 33 brought down, so 0 is written in the root and afte 4, and a new period, 47, is brought down. Next, 40 is understood to be 40 tens or 400. This 400 is contained in 3347 eight times. When the decima period is brought down, the decimal point is inserted in the root.

3. 14,356,521. **4.** 33,790,969. **5.** 16,803.9369.

Solve the following, getting three decimal places in the root, and prove answers by squaring the root and adding in remainde thus getting given number.

6. 2.5. **7.** $\frac{1}{70}$. **8.** .008. **9.** .1.

a. Every sequence of figures has two square roots due to the position of the decimal point. Thus $\sqrt{25} = 5$; $\sqrt{2.5} = 1.6^-$. This shows importance of pointing off number into periods correctly, commencing at decimal point. Notice $\sqrt{.5}$ must be written $\sqrt{.50}$ to get results correct. It is important to check the square root of decimals by multiplication, especially when first beginning the subject.

10. Make a rule for extracting square root of arithmetical numbers, explaining carefully exceptional cases.

11. 2. **12.** .056. **13.** $\frac{2}{7}$. **14.** .00003.

Solve following mentally, getting one decimal place in root:

15. 2. **16.** 3. **17.** .5. **18.** 209.

***173. Extraction of Cube Root of Algebraic Quantities.** To derive a process for the extraction of the cube root of polynomials from the formula:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + (3a^2 + 3ab + b^2)b.$$

SOLUTION

$$\begin{array}{r|l} & a^3 + 3a^2b + 3ab^2 + b^3 \;\big|\; \underline{a+b} \\ & a^3 \\ \hline 3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\ & \underline{3a^2b + 3ab^2 + b^3} \end{array}$$

How is first term of root found? How is trial divisor $3a^2$ found from first term of root? How is second term of root found? What two terms are added (annexed) to the trial divisor to make complete divisor? Explain precisely how they are obtained. How is process completed? Is it important that the terms be arranged before beginning?

---

*This subject is often omitted. It is not required for college entrance.

1. $8x^6 + 66x^4 - 63x^3 + 33x^2 - 9x + 1 - 36x^5$.

Solution

$$\begin{array}{r|l l}
 & 8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1 & \underline{|2x^2 - 3x + 1} \\
 & \underline{8x^6} & \\
12x^4 - 18x^3 + 9x^2 & -36x^5 + 66x^4 - 63x^3 & \\
 & \underline{-36x^5 + 54x^4 - 27x^3} & \\
12x^4 - 36x^3 + 27x^2 & 12x^4 - 36x^3 + 33x^2 - 9x + 1 & \\
6x^2 - 9x & & \\
+1 & & \\
\hline
12x^4 - 36x^3 + 33x^2 - 9x + 1 & \underline{12x^4 - 36x^3 + 33x^2 - 9x + 1} &
\end{array}$$

Explanation. 1st trial divisor $= 3(2x^2)^2$.

2d trial divisor $= 3(2x^2 - 3x)^2$.

$3(2x^2 - 3x) \times 1 = 6x^2 - 9x$.

$1^2 = 1$.

2. $x^3 - 24x^2 + 192x - 512$.

3. $x^6 - 3x^5y + 6x^4y^2 - 7x^3y^3 + 6x^2y^4 - 3xy^5 + y^6$.

4. $8x^6 + 48cx^5 + 60c^2x^2 - 80c^3x^3 - 90c^4x^2 + 108c^5x - 27c^6$.

5. Make a rule for finding the cube root of algebraical quantities.

6. $39x^4 - 99x^3 - 9x^5 + x^6 + 64 - 144x + 156x^2$.

7. Find the first three terms of the cube root of $1 - x$. Prove by cubing root (§ 167) and adding remainder.

*174. Extraction of the Cube Root of Arithmetically Expressed Numbers.

1. Extract cube root of 405,224.

Solution. Since 405,224 lies between 343,000, which is the cube of 70, and 512,000, which is the cube of 80, the cube root of 405,224 lies between 7 tens and 8 tens. In general, counting from unit's order, each set of three figures or "period" in the number gives one figure in the root, except that the left-hand period may have three or two figures or only one figure. Thus, the cube root of 1 728 is 12.

---

* This subject is often omitted. It is not required for college entrance.

Solution with Explanation

$$\begin{array}{rrl}
 & & t+u \\
 & 405'224 & \underline{|\,70+4} \\
t^3 = & \underline{343\,000} & \\
3\,t^2 = 14700 & |\;62\,224 & \\
3\,tu = \quad 840 & | & \\
u^2 = \quad \underline{16} & | & \\
15556 & |\;\underline{62\,224} &
\end{array}$$

Regular form of Solutio

$$\begin{array}{rrl}
 & 405'224 & \underline{|\,74} \\
 & \underline{343} & \\
3\times70^2 = 14700 & |\;62\,224 & \\
3\times70\times4 = \quad 840 & | & \\
4^2 = \quad \underline{16} & | & \\
15556 & |\;\underline{62\,224} &
\end{array}$$

2. Extract the cube root of 232435.510 to one decimal plac and prove answer.

Solution.

$$\begin{array}{rrl}
 & 232'435.'510 & \underline{|\,61.4} \\
 & \underline{216} & \\
10800 & |\;16\,435 & \\
180 & | & \\
\underline{1} & | & \\
10981 & |\;\underline{10\,981} & \\
1116300 & |\;5454.510 & \\
7320 & | & \\
\underline{16} & | & \\
1123636 & |\;\underline{4494.544} & \\
 & 959.966 &
\end{array}$$

Explanation

$3\,t^2 = 3 \times 60^2 = 10800,$

$3\,tu = 3 \times 60 \times 1 = 180,$

$u^2 = 1^2 = 1,$

$3\,t^2 = 3 \times 610^2 = 1116300,$

$3\,tu = 3 \times 610 \times 4 = 7320,$

$u^2 = 4^2 = 16.$

Check. $(61.4)^3 + 959.966 = 232435.510.$

The answer to the nearest tenth is 61.5, since the next figure of the roo after 4 would be more than 5.

3. 12,167. 4. 12,812.904. 5. 167.284151.

In the following get 2 places in the root and verify:

6. 3. 7. 0.2. 8. $\frac{1}{254}$.

9. Write out a rule for cube root, including peculiarities.

In following get answer to nearest tenth mentally:

10. 10. 11. 37. 12. 150.

175. Symmetry in Algebra. A quantity is **symmetrical** if the letters in it can change places without changing its value.

Thus, $a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4$ is symmetrical, as is the sum of a and b raised to any integral power, since a and b can interchange without altering the value of the expression.

Similarly, $x^2 + y^2$, $x^2 \pm xy + y^2$, $\dfrac{a^2+b^2}{2\,ab}$, etc., are all symmetrical.

III. ABRIDGED MULTIPLICATIONS AND DIVISIONS

***176. Abridged Multiplication.** Every one who has used the regular process of multiplication of decimals has felt at times that a portion of the work was wasted.

In the Arabic language, as in the Hebrew, the writing extends from right to left. This accounts for calculations in the Arabic notation proceeding from right to left instead of from left to right as in our reading. But there is no reason why we cannot begin multiplying by the highest order instead of by the lowest.†

1. Multiply 172.181 by 163.56, getting the product correct to one decimal place.

SOLUTION

```
  172.ˎ1ˎ8ˎ1
  163.56
 ──────────
17218.1
10330.86
  516.54
   86.09
   10.33
 ──────────
28161.9
```

EXPLANATION. The multiplier is set under the multiplicand with one decimal point under the other. We multiply first by 1, which is here 100. This moves the decimal point in the product two places to the right of where it now is in multiplicand, bringing the right-hand figure of the product in tenths' place. Next we multiply by 6 (60), starting this product one place to the right of the product just set down.

Now, to get tenths in the answer correct, hundredths should be kept. But, multiplying by 3, one gets thousandths, which figure need not be written down. We therefore insert a little mark before the last figure in the multiplicand, and say, 3 × ˎ1 = .3, giving nothing to carry. Then, 3 × 8 = 24; the 4 is set down under 6, and from there on it is ordinary multiplication. When we multiply by the next figure, 5, another figure is cut off in the multiplicand, and we say, 5 × ˎ8 = 4.0, giving 4 to carry; 5 × 1 = 5; 5 and 4 to carry are 9. This 9 is set down under 4 in hundredths' order; and so on.

In adding we do not write the hundredths in the answer, but merely find the number to the nearest ten to carry from the last column.

2. Rule for abridged multiplication.

(1) *Set the multiplier under the multiplicand as if for addition.*

---

* This subject is treated here because it is not found in many arithmetics. Pupils should accustom themselves to performing multiplication of decimals as here explained, because time is saved and the process is more rational than the usual one.

† In the "Encyclopædia Britannica" (11th Edition) under the title *Arithmetic*, will be found a discussion of this question of the order in multiplication.

(2) *Multiply first by the left-hand figure of the multiplier. If the first multiplier figure is units, set the right-hand figure of the partial product directly under the right-hand figure of the multiplicand. If the first figure is not units, move the product to the left or right as many places as the figure is to the left or right of units, and proceed in the same way with the other multiplier figures.*

(3) *One more decimal place in the product should be kept than is asked for in the answer. As soon as a multiplier figure gives an order lower than the lowest to be kept, the right-hand figure of the multiplicand is cut off by a little mark, and when this figure is multiplied by the multiplier figure, the product is not set down, but the tens, to the nearest unit, are carried, and added with the product of the next figure of the multiplicand by this multiplier figure.*

In the next partial multiplication, another figure is cut off at the right of the multiplicand, and the multiplication proceeds in the same way; and so on.

(4) *In adding, the sum in the right-hand column is not set down, but the nearest number of tens is carried.*

3. Multiply 6.7421 by 4.7825, getting tenths correct. Check by ordinary multiplication.

4. Multiply 6.9321 by 4.7825, getting hundredths correct. Check.

5. Multiply 65.324 by 63.412, getting tenths correct. Check.

6. Multiply 264.72 by 34.621, getting tenths correct. Check.

7. If the pupil will now compare the usual solutions with the abridged solutions, he will see that he can draw a vertical line through the usual solution cutting off one part and having left the abridged solution, but with its right column somewhat changed by carrying.

8. Find the compound interest, correct to the nearest cent, on $729.38 for four years at 6% if the interest on $1 as given in a table is $.262477.

9. Find the compound interest amount on $867.93 for 7 yr. 6 mo. if the rate is 5% and the interest is compounded semi-

annually, if the compound interest table gives $1.448298 as the amount on $1 for the given time at the given rate.

10. Find the cost correct to cents of 41,654 ft. of lumber at $18.75 per M.

11. Find the cost of 6347 lb. of coal at $7.25 a ton.

177. Abridged Division. 1. A stock that pays 7.5% annual dividends, that is, $7.50 per share of $100, costs, including brokerage, $164.375. Find the rate of interest realized correct to four decimal places.

SOLUTION

```
                .0456
16‸4‸3‸75)7500.00
          6575
           925
           822
           103
            98
```

EXPLANATION. The decimal point is moved three places to the right in both divisor and dividend, which is equivalent to multiplying each by 1000, thus leaving the quotient unaltered. This makes the divisor a whole number, places the decimal point in the quotient immediately over that in the dividend, and the first significant figure of the quotient over the last figure of the dividend used to get it when all figures of the divisor are used.

The order of the first significant figure in the quotient is thus hundredths. his leaves two figures of the quotient still to be found. If we wish 2 figures nly to appear in the divisor at the end of the division, and one figure is cut ff from the divisor each time as we get the two remaining figures of the uotient, then four figures will be all we shall need in the divisor when e begin the division. Consequently 75 is cut off, thus, ‸75, before beinning to divide.

The operation of dividing proceeds as follows: 4 × .7 = 2.8, giving 3 to e nearest unit to carry; then 4 × 3 + 3 = 15, the 5 being set down and 1 rried; then the operation continues as usual. In this way we get 6575.

To get the next figure of the quotient, the 3 in the divisor is now cut off y a mark. Then we say 5 × .3 = 1.5, giving 2 to carry; 5 × 4 + 2 = 22, 2 eing set down and 2 carried; in this way we get 822. In the last operaon, 4 is cut off, and we begin by saying 6 × .4 = 2.4, which gives 2 to carry; en 6 × 6 + 2 = 38, or 8 down and 3 to carry; 6 × 1 + 3 = 9.

2. **Rule for Abridged Division.** (1) *Make the divisor a whole umber without zeros at its right, by moving the decimal point. fove the decimal point in the dividend the same number of places nd in the same direction as in the divisor.*

(2) *Place the decimal point in the quotient immediately over tha in the dividend, and the first significant figure in the quotient ove the last figure of the dividend used to get it if all the figures of th divisor were used.*

(3) *To the number of figures in the quotient yet to be foun add 2. This gives the number of figures needed in the diviso Cut off with mark any excess over this at the right of the diviso If there is no such excess at the beginning of the operation, test fo such excess as you proceed, using the rule just given.*

(4) *Each time multiply the left-hand figure of those cut off by th figure of the quotient and carry the tens of the product to the neares unit, but do not set down the units. In each successive divisio thereafter cut off a new figure at the right of the divisor.*

3. Divide 163.27 by 48.613, getting three decimal places in th quotient.

4. Divide 267.46 by 23,740, getting three decimal places i quotient.

5. Find the present worth of $258.19 due in 9 mo. and 14 d at 6 %, that is, divide 258.19 by 1.04733^+, getting answer correc to cents.

6. A man bought a stock of goods for $12,674.29 and sold i for $14,695.18. What per cent, to the nearest tenth, did he gain

7. How many bushels, to the nearest hundredth, are there i 66,374 cu. in., if 1 bu. contains 2150.42 cu. in.?

8. What principal, to the nearest cent, will amount to $756.2 in 2 yr. 8 mo. 17 da., at 6 %?

9. What is the diameter of a circle, to the nearest tenth, whos circumference is 65.8 in., if π is taken equal to 3.14? If 65.832 is circumference, and π is taken equal to 3.1416?

10. What is the width of a rectangle whose length is 16.8 rd and whose area is 818.19 sq. yd., getting answer to nearest tent of a yard?

V. PRACTICAL APPLICATION OF INVOLUTION, EVOLUTION, AND ABRIDGED MULTIPLICATION AND DIVISION

178. Solution of Problems by the Use of Involution, Evolution, ultiplication, and Division. Formulas.

1. Calculate in the quickest way $2^2 \times 3^2 + 2 \times 3^3 + 2^3 \times 3^3$ $+ 2 \times 3^4$ by taking out the common factor.

2. Calculate $18^2 + 15^2 + 6^3 + 9^3 + 36^2$ by changing each quanity into the product of its prime factors with exponents and then king out the common factor. Thus, $18^2 = (2 \times 3^2)^2 = 2^2 \times 3^4$; etc.

3. Write as the product of their prime factors raised to owers, each of the following: 54^3; 125^2; 24^4; 108^3; 36^5.

4. $(a+b+c)^2 + (a-b-c)^2 = ?$ See § 68.

5. $(x+y-z)^2 - (x-y+z)^2 = ?$ 6. $(a-b)^2(b-a)^3 = ?$

7. Find the area of a square whose side is 6.05 ft., correct to nths. See § 63.

8. Find the volume of a cube whose edge is $8\frac{1}{8}$ in., correct to vo decimal places. § 166, 11.

9. Find the area, correct to tenths, of a circle whose radius is $1\frac{1}{2}$ in. from the formula $a = \pi r^2$, using $\pi = 3.1416$. (See § 88 r square of number $+ \frac{1}{2}$.)

10. Find the volume, correct to hundredths, of a sphere whose iameter is 4.02 from the formula $v = \frac{1}{6}\pi d^3$.

11. Find the surface, correct to hundredths, of a sphere whose dius is $3\frac{1}{2}$ m., from the formula $s = 4\pi r^2$.

12. Find the volume correct to hundredths of a cylinder whose titude is 3.875 ft. and radius of base 2.268 ft., from the formula $= \pi r^2 h$, in which r is the number of feet in the radius of base and the number of feet in the altitude.

13. Find the volume correct to hundredths of a cone whose titude a is 20.5 cm. and radius of its base 144 cm., from the rmula $v = \frac{1}{3}\pi a r^2$.

14. Find the area correct to hundredths of the triangle whos three sides a, b, c are respectively 8, 9, 13 in. from the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{(See § 89.)}$$

in which

$$s = \tfrac{1}{2}(a+b+c).$$

Solution. $s = \frac{1}{2}(8+9+13) = 15$; $s - a = 7$; $s - b = 6$; $s - c = 2$.

Then $A = \sqrt{15 \times 7 \times 6 \times 2} = \sqrt{1260} = 35.5^{-}$. *Ans.*

Check. $35.5^2 = 1260.25$.

15. Find the areas correct to hundredths of the followin triangles in same way: $a = 10$, $b = 12$, $c = 14$; $a = 1.6$, $b = 1.9$ $c = 2.1$; $a = 346$, $b = 579$, $c = 722$; $a = .06$, $b = .053$, $c = .019$

16. Find the hypotenuse correct to hundredths of a right tr angle whose two legs are 6.21 ft. and 9.42 ft. from the formul $h = \sqrt{a^2 + b^2}$.

17. Find one leg of a right triangle whose hypotenuse is 7 in. and other leg is $4\frac{1}{4}$ in. from the formula $a = \sqrt{h^2 - b^2}$.

18. Find the time of falling correct to tenths of a second fror the formula $t = \sqrt{\dfrac{2s}{g}}$, knowing the distance to be 168.2 ft. an $g = 32.2$ ft.

19. Find the surface correct to two decimal places of a cub whose volume is 168.2 cu. ft.

20. Find the surface of a sphere whose volume v is 19 cu. f from the formula $s = \sqrt[3]{36\,\pi v^2}$.

21. The volume v of a spherical segment or part of sphere in cluded between two circles in parallel planes whose radii are and r' and whose altitude h is the perpendicular distance betwee the parallel planes is given by the formula

$$v = \frac{h}{2}\left(\pi r^2 + \pi r'^2\right) + \frac{\pi h^3}{6}.$$

Find v when $r = 6.1$, $r' = 8.2$, and $h = 1.08$. (See § 166, 11.)

22. Find the length of a perpendicular from the right angle to the hypotenuse of a right triangle if it divides the hypotenuse which is 10 ft. long into segments one of which is .5 ft. longer than the other.

Suggestion. Solve first for x and y, the segments of the hypotenuse. See § 51.

23. By measurement the distances on a line drawn from a point outside a circle to it are 12.7 ft. to the nearest point of the circumference and 19.6 ft. to the farther point on the circumference. What is the length on the tangent to the circle from the given point measured from the point of tangency to the given point?

24. If AB is the chord of a circle whose radius is R, and AD is the chord of half the arc AB, then, by geometry,

$$AD = \sqrt{2R - R\sqrt{4R^2 - \overline{AB}^2}}.$$

If the radius of a circle is 8 ft. and the chord AB is 2 ft., find AD.

25. If AB of preceding Exercise is one side of a regular hexagon inscribed in a circle whose radius is 1 ft., then AD is one side of a regular inscribed 12-agon. Calculate the *perimeter* of the egular 12-agon after first calculating one of its sides by means of he formula in Ex. 24. (See § 51, 21.)

26. Using the value of AD found in Ex. 25 calculate one side of a regular 24-agon inscribed in a circle whose radius is 1 ft., and hen find its perimeter.

27. Explain, now, how the formula of Ex. 24 can be used to ind an approximate value of π.

CHAPTER VII

FRACTIONAL EXPONENT QUANTITIES AND RADICALS.[*]

I. FRACTIONAL EXPONENT EXPRESSIONS

179. Formulas Giving the Laws for Integral Exponents.

1. Addition and subtraction formulas:

$$ax^m + bx^m = (a + b)x^m; \quad ax^m - bx^m = (a - b)x^m. \quad (\S\,35.)$$

2. Multiplication formula: $x^m \times x^n = x^{m+n}$. (§ 42.)

3. Division formula: $x^m \div x^n = x^{m-n}$; $\frac{x^m}{x^n} = x^{m-n}$. (§ 48.)

4. Power formulas: $(x^m)^n = x^{mn}$; $(xyz)^m = x^m y^m z^m$. (§ 164.)

5. Root formulas: $\sqrt[n]{x^m} = x^{\frac{m}{n}}$; $\sqrt[n]{x^m y^p} = x^{\frac{m}{n}} y^{\frac{p}{n}}$. (§ 169.)

180. Meaning of Fractional Exponents. Integral exponents we defined in § 11. We are now to determine a meaning for fra tional exponents. Such definition will naturally have to agr with the definition and formulas for integral exponents, since i tegral exponents are merely special cases of fractional exponen having the denominator 1.

If we let the laws of § 164 and § 41 hold true for fractional e ponents, we have:

$$(x^{\frac{1}{2}})^2 = x^{\frac{2}{2}} = x; \quad x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x.$$

Thus, $x^{\frac{1}{2}}$ *is the square root of* x, since it is *one* of two equal f tors whose product is x. (See § 13.)

Again, $(x^{\frac{1}{3}})^3 = x^{\frac{3}{3}} = x$; $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x^1 = x$.

---

[*] Throughout this chapter many problems can be solved mentally and sho be so solved.

Thus, $x^{\frac{1}{3}}$ *is the cube root of* x, since it is *one* of three equal factors whose product is x.

In general, $(x^{\frac{m}{n}})^n = x^{\frac{mn}{n}} = x^m$;

$$x^{\frac{m}{n}} \times x^{\frac{m}{n}} \times x^{\frac{m}{n}} \times \cdots \text{ to } n \text{ factors} = x^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\text{to } n \text{ terms}} = x^{\frac{mn}{n}} = x^m.$$

Thus, $x^{\frac{m}{n}}$ is the *nth* root of x^m, since it is *one* of n equal factors whose product is x^m.

We are therefore led to attach the following meaning to a fractional exponent:

The numerator of a fractional exponent denotes the power to which the quantity of which it is the exponent is to be raised, and the denominator denotes the root to be taken of this result.

The student will see this more clearly from the following examples; $\sqrt{a^4} = a^{\frac{4}{2}} = a^2$; $\sqrt[3]{x^6} = x^{\frac{6}{3}} = x^2$; $\sqrt[4]{b^{12}} = b^{\frac{12}{4}} = b^3$; then $\sqrt[3]{a^7} = a^{\frac{7}{3}}$, and $a^{\frac{7}{3}}$ means the cube root of the 7th power of a.

181. Exercise dealing with Quantities having Fractional Exponents.

1. Calculate $8^{\frac{2}{3}}$.

Solution. $8^2 = 64$; $\sqrt[3]{64} = 4$. *Ans.* Or, $\sqrt[3]{8} = 2$; $2^2 = 4$. *Ans.*

Evidently it is easier to extract the root *first.*

Another Solution. $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$. *Ans.*

Here the number is first expressed as the power of a prime.

a. To extract the root of a fraction extract the root of each term.

$$\text{Thus, } \sqrt{\frac{4}{9}} = \frac{2}{3}. \quad \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}.$$

2. Calculate: $4^{\frac{1}{2}}$; $16^{\frac{1}{4}}$; $27^{\frac{2}{3}}$; $4^{\frac{3}{2}}$; $9^{\frac{3}{2}}$; $(\frac{1}{8})^{\frac{2}{3}}$; $(-\frac{1}{27})^{\frac{2}{3}}$; $(\frac{8}{27})^{\frac{4}{3}}$; $(a^m)^{\frac{1}{m}}$; $(x^{pq})^{\frac{1}{q}}$.

3. Calculate: $16^{\frac{1}{4}}$; $16^{\frac{1}{2}}$; $49^{\frac{3}{2}}$; $(36)^{\frac{5}{2}}$; $(\frac{1}{4})^{\frac{7}{2}}$.

4. $(9m^4 - 30m^2n + 25n^2)^{\frac{1}{2}} = ?$

5. $(a^3 - 6a^2b + 12ab^2 - 8b^3)^{\frac{1}{3}} = ?$

6. Express with fractional exponents: $\sqrt[5]{x^4}$ (*Ans.* $x^{\frac{4}{5}}$); $\sqrt[3]{a^2}$; $\sqrt[6]{m^{11}}$; $\sqrt[m]{y^n}$; $\sqrt[3]{m^2}$.

7. Express with radical signs: $x^{\frac{4}{5}}$; $a^{\frac{3}{4}}$; $p^{\frac{1}{2}}$; $3z^{\frac{1}{3}}$.

8. Calculate $243^{\frac{2}{5}}$; $36^{\frac{3}{2}}$; $9^{\frac{5}{2}}$; $81^{\frac{3}{4}}$.

182. Principles underlying Operations with Quantities containing Fractional Exponents.

We know that $(abc)^2 = a^2b^2c^2$; $(abc)^3 = a^3b^3c^3$; etc. The question arises, is it likewise true that

$$(abc)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}};\ (abc)^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}};\ \text{etc.?}$$

Or, stated generally, is it true that

$$(1) \qquad (abc\cdots)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots?$$

Now we can test the truth of this equation by raising both sides to the nth power by the power axiom (§ 57). We have

$$[(abc\cdots)^{\frac{1}{n}}]^n = abc\cdots. \text{ (By the definitions of root and power.)}$$

and

$$(a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots)^n = (a^{\frac{1}{n}})^n(b^{\frac{1}{n}})^n(c^{\frac{1}{n}})^n\cdots$$
$$= abc\cdots. \text{ (By the definitions of root and power.)}$$

Thus, we see that raising the two members of equation (1) to the same power gives equals. This, however, does not *prove* that $(abc\cdots)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots$, since in algebra unequals raised to the same power may give equals. For example, $(-2)^2 = +4$, and $(+2)^2 = +4$; but this does not prove that -2 equals $+2$, for they are *not* equal.

If, however, we limit the roots considered to arithmetical values, that is, to positive numbers, then it is always true that if two numbers raised to the same power give equals, they are equal.

Hence, with the limitation in meaning stated, we have

$$(abc\cdots)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots.$$

Changing this formula into a principle, we obtain:

1. **Fundamental Principle.** *The arithmetical root of a product is equal to the product of the same arithmetical roots of the several factors, and conversely.*

By means of this principle any **factor** within a sign denoting a root may be removed outside the sign provided the desired root of the factor can be extracted.

For example, $12^{\frac{1}{2}} = (4 \times 3)^{\frac{1}{2}} = 4^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 2(3)^{\frac{1}{2}}$.

By means of this principle also like roots of quantities can be multiplied together. Thus, $6^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 30^{\frac{1}{2}}$.

2. *If both terms of a fractional exponent are multiplied by the same number, the value of the quantity is not changed.*

This principle holds, likewise, only for arithmetical roots.

For, while $9^{\frac{1}{1}} = 9$, $9^{\frac{2}{2}}$ may equal either $+9$ or -9, since the square root (denoted by the denominator 2 of the exponent) may be either positive or negative.

3. *A quantity with a fractional exponent may have its value calculated either by raising to the power first and extracting the root afterwards, or vice versa.* See Ex. 1, § 181.

This principle also holds true only for arithmetical roots.

Thus, $25^{\frac{2}{2}} = (625)^{\frac{1}{2}} = \pm 25$, while $(25^{\frac{1}{2}})^2 = (\pm 5)^2 = +25$ only.

183. Proofs based on principles of § 182 to show that same laws govern the use of fractional exponents as governed integral ones.

1. *Proof that in multiplication the exponents of like factors are added.*

Let $x^{\frac{m}{n}}$ and $x^{\frac{p}{q}}$ be any two quantities with fractional exponents, x being any quantity, and m, n, p, q being integral numbers.

Then,
$$
\begin{aligned}
x^{\frac{m}{n}} \times x^{\frac{p}{q}} &= x^{\frac{mq}{nq}} \times x^{\frac{np}{nq}} && (\text{§ 182, 2.})\\
&= (x^{mq})^{\frac{1}{nq}} \times (x^{np})^{\frac{1}{nq}} && (\text{§ 182, 3.})\\
&= (x^{mq} \times x^{np})^{\frac{1}{nq}} && (\text{§ 182, 1.})\\
&= (x^{mq+np})^{\frac{1}{nq}} && (\text{§ 42.})\\
&= x^{\frac{mq+np}{nq}}. && (\text{§ 182, 3.})
\end{aligned}
$$

But $\frac{mq+np}{nq}$ is the **sum** of the exponents $\frac{m}{n}$ and $\frac{p}{q}$.

Hence the rule to add the exponents of the same letter in the factors for the exponent of this letter in the product holds for fractional as well as for integral exponents.

2. Derive the corresponding rule for division. (See § 48.)

3. *Proof that in raising a quantity to a power the exponent, of the quantity is multiplied by the index of the power for the exponent of the quantity in the result.*

Let $a^{\frac{m}{n}}$ be the given quantity and $\frac{p}{q}$ be the exponent, a being any quantity and m, n, p, q being integral numbers.

Then $(a^{\frac{m}{n}})^{\frac{p}{q}}=([(a^m)^{\frac{1}{n}}]^{\frac{1}{q}})^p$ (By § 182, 3.)

$=[(a^m)^{\frac{1}{nq}}]^p$ (Since the qth root of the nth root means the nqth root by the definition of root, § 13, the nqth root being one of q equal factors of *one* of the n equal factors of a^m.)

$=[(a^m)^p]^{\frac{1}{nq}}$ (By § 182, 3.)

$=(a^{mp})^{\frac{1}{nq}}$ (Since the pth power of the mth power is the mpth power, § 164.)

$=a^{\frac{mp}{nq}}$. (By § 182, 3.)

But $\frac{mp}{nq}$ is the *product* of the exponent of the quantity and the exponent of the power.

Hence the rule to multiply the exponent of the quantity by the exponent of the power holds for fractional as well as integral exponents.

184. Exercise in performing Operations involving Quantities having Fractional Exponents.

Work mentally the first ten and as many more as possible.

1. Simplify $3a^{\frac{2}{3}} \times -5a^{\frac{3}{5}}$. *Ans.* $-15a^{\frac{19}{15}}$.

2. $9a^{\frac{7}{6}}b \times 3a^{\frac{5}{6}}b^{\frac{2}{3}}$. 3. $3m^{\frac{2}{3}}n^{\frac{4}{5}} \times 6m^{\frac{1}{6}}n^{\frac{1}{5}}p$. 4. $a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{5}{6}} \times a^{\frac{1}{2}}bc^{\frac{1}{3}}$.

5. $a^{\frac{m}{n}} \times a^{\frac{p}{q}} \times a^{\frac{r}{s}}$. 6. $12x^{\frac{1}{3}}y^{\frac{4}{3}}z^{\frac{1}{2}} \div -2x^{\frac{1}{5}}y^{\frac{2}{5}}$. 7. $6x^{\frac{a}{b}} \div -2x^{\frac{c}{d}}$.

8. $(3a^{\frac{1}{2}})^3$. **Remark.** Notice that just as $(ab)^2 = a^2b^2$, so here $(3a^{\frac{1}{2}})^3 = 3^3 \times (a^{\frac{1}{2}})^3$. (See § 182).

9. $(9a^{\frac{5}{6}})^{\frac{1}{2}}$; $(8a^{\frac{6}{7}})^{\frac{1}{3}}$. 10. $(a^{\frac{1}{3}})^{\frac{2}{5}}$; $(-4a^{\frac{1}{5}})^2$; $(-27a^{\frac{2}{3}})^{\frac{1}{3}}$.

11. $(x^{\frac{3}{4}} - x^{\frac{1}{2}} + 2x^{\frac{1}{4}} + 3)(x^{\frac{1}{4}} - 2)$. (§ 39, III.)

12. $(x - y) \div (x^{\frac{1}{3}} - y^{\frac{1}{3}})$; $(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1)(a^{\frac{1}{3}} - 1)$.

13. $(x^{\frac{1}{4}} - 6y^{\frac{1}{3}})(x^{\frac{1}{4}} + 5y^{\frac{1}{3}})$. (§ 66.)

14. $(x^{\frac{1}{4}} + y^{\frac{1}{4}})^2$; $(a^{\frac{1}{3}} - 5b^{\frac{1}{3}})^3$; $(3m^{\frac{1}{5}} - 2n^{\frac{2}{5}})^4$.

15. From the sum of $5ax^{\frac{1}{2}} - (x + y)^{\frac{1}{2}} + (a - b)^{\frac{1}{3}}$ and $-7ax$ $+ 2(x + y)^{\frac{1}{2}} - 3(a - b)^{\frac{1}{3}}$ take $3ax^{\frac{1}{2}} + 4(x + y)^{\frac{1}{2}} - 5(a - b)^{\frac{1}{3}}$.

16. Calculate $36^{1\frac{1}{2}} + 4^{3\frac{1}{2}} - 100^{0.5} - 81^{0.75}$.

17. Find the product of $\left(\frac{ay}{x}\right)^{\frac{1}{2}}$, $\left(\frac{bx}{y^2}\right)^{\frac{1}{3}}$, and $\left(\frac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$.

18. Divide $x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.

19. $(a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}})^2 = ?$ $(a^{\frac{3n}{2}} + b^{\frac{3n}{2}}) \div (a^{\frac{n}{2}} + b^{\frac{n}{2}})$.

20. $\sqrt{(1 + 4x^{\frac{1}{3}} - 2x^{\frac{2}{3}} - 4x + 25x^{\frac{4}{3}} - 24x^{\frac{5}{3}} + 16x^2)} = ?$

21. Factor $a^{\frac{3}{2}} - 1$ (§ 70); $a^2 + ab + b^2$ (§ 80, 16).

22. $(x^{\frac{6}{5}} - 9x + 33x^{\frac{4}{5}} - 63x^{\frac{3}{5}} + 66x^{\frac{2}{5}} - 36x^{\frac{1}{5}} + 8)^{\frac{1}{3}} = ?$

23. Expand $(x^{\frac{2}{5}} - 4)(x^{\frac{2}{5}} + 5)$.

24. Expand $(x^{\frac{2}{3}} + 4y^{\frac{2}{3}})(x^{\frac{2}{3}} - 4y^{\frac{2}{3}})$.

25. Expand $(x^{\frac{n}{3}} - 2y^{\frac{2p}{3}})^3$.

26. Expand $(x^{\frac{1}{2}} - 2y^{\frac{1}{2}} - 3z^{\frac{1}{2}})^2$.

185. Meaning of Zero and Negative Exponents.

1. Zero exponents arise naturally in division when a letter has the same exponent in both divisor and dividend.

Thus, $x^m \div x^m = x^{m-m} = x^0$. (§ 48.)

But $x^m \div x^m = 1$.

Hence, if the rule for subtracting exponents is to continue to hold, we must take $x^0 = 1$. (§ 57, 8.) We have then this important result:

2. **Theorem concerning 0 Exponent.** *Any finite quantity with exponent* 0 *is equal to* 1.

The meaning of this theorem is understood better if we think of x^0 as being associated with other factors.

For example, $\dfrac{6x^2yz}{x^2uv} = \dfrac{6x^0yz}{uv} = \dfrac{6yz}{uv}$.

The exponent 0 shows that x is used *no* times as a factor of the product, or has dropped out and so does not affect the product of the other factors. x^0 cannot have the value 0, for that would make the whole expression 0, and it is *not* 0.

3. The meaning to attach to negative exponents can be obtained from that for zero exponent. We have

$x^m \times x^{-m} = x^{m+(-m)} = x^0 = 1$. (As just shown).

Or $x^m \times x^{-m} = 1$. (Things equal to same thing are equal to each other.)

Then $x^{-m} = \dfrac{1}{x^m}$ (Division Axiom).

Thus, we are led to take a quantity with a negative exponent to mean the *reciprocal* of the same quantity with a positive exponent.

If $b^{-2} = \dfrac{1}{b^2}$, and $c^{-3} = \dfrac{1}{c^3}$, as just laid down,

$$\frac{ab^{-2}}{c^{-3}d} = \frac{a \times \frac{1}{b^2}}{\frac{1}{c^3} \times d} = \frac{\frac{a}{b^2}}{\frac{d}{c^3}} = \frac{ac^3}{b^2d}. \quad (\text{§ } 106.)$$

Here we see b^{-2} in the numerator of the first fraction become b^2 in the denominator of the last, and c^{-3} in the denominator of the first fraction become c^3 in the numerator of the last. In this way we get the following theorem:

4. **Theorem concerning Negative Exponents.** *Any factor can be transferred from the numerator to the denominator or from the denominator to the numerator of a fraction provided the sign of its exponent is changed.*

186. Exercise in Using Zero and Negative Exponents.

Write the values of the following:

1. 10^0. 2. 1000^0. 3. a^0b^0c. 4. $\frac{4x^2y^3}{x^2y}$.

Express the following with positive exponents:

5. $2a^{-5}b$. 6. $m^{-1}n^{-2}$. 7. $5a^{\frac{1}{2}}c^{-\frac{1}{3}}$. 8. $10a^{-\frac{5}{4}}b^{\frac{2}{3}}$.

Write the following fractions without denominators:

9. $\frac{2}{x^{-3}}$. 10. $\frac{1}{a^{-2}b^3}$. 11. $\frac{a^2m^4}{2b^{\frac{1}{3}}}$. 12. $\frac{5}{3^{-1}c^{\frac{2}{3}}d^{-\frac{1}{4}}}$.

(*a*) When no denominator is written, 1, of course, is understood. When all the factors of a numerator are transferred to the denominator (as in Ex. 6), the factor 1 is understood to remain in the numerator

187. Rules giving the Laws for Exponents whether Integral, Fractional, or Negative.

1. *In addition, add the coefficients of similar quantities. The exponents in the literal part remain unchanged.*

2. *In multiplication, add the exponents of the same quantity in the factors for the exponent of this quantity in the product.*

3. *In division, subtract the exponent of a factor in the divisor (denominator) from the exponent of the same factor in the dividend (numerator) for the exponent of this quantity in the quotient.*

4. *In raising to powers, multiply the exponent of a factor quantity by the index of the power to which it is to be raised for the exponent of this quantity in the answer.*

5. *In extracting roots, divide the exponent of a factor quantity by the index of the root for the exponent of this quantity in the answer.*

6. *Any finite quantity with exponent* 0 *is equal to* 1.

7. *Any factor of the numerator or denominator of a fraction can be transferred to the other term of the division by changing the sign of its exponent.*

188. General Exercise in the Simplification of Quantities involving Fractional and Negative Exponents. Answers are to have only positive exponents. Solve as many as possible mentally.

In solving numerical exercises write quantities as powers of prime factors, and then, after simplifying, change all negative exponents to positive ones.

Thus, $$8^{-\frac{2}{3}} = (2^3)^{-\frac{2}{3}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}.$$

1. $16^{\frac{3}{4}}$.
2. $81^{\frac{1}{4}}$.
3. $100^{-\frac{1}{2}}$.
4. $625^{0.75}$.
5. $x^{\frac{2}{3}} \times x^{-\frac{1}{2}}$.
6. $m^{\frac{2}{3}} \times m^{\frac{3}{2}}$.
7. $(a^{\frac{1}{3}})^{\frac{3}{5}}$.
8. $16^{-\frac{1}{4}}$.
9. $a \times a^{-\frac{2}{5}}$.
10. $(a^4 b^{-\frac{2}{3}})^{\frac{3}{2}}$.
11. $100^{-\frac{5}{2}}$.
12. $4\,a^{\frac{1}{3}} \div 2\,a^{-\frac{5}{6}}$.
13. $(\frac{4}{25}\,a^2)^{-\frac{3}{2}}$.
14. $(\frac{27}{125})^{-\frac{2}{3}}$.
15. $a^{\frac{1}{3}} \times a^{-\frac{3}{4}} \times a^{\frac{5}{12}}$.
16. $((a^{-2})^2)^{\frac{3}{4}}$.
17. $(xy)^{x+y} \div x^y y^x$.
18. $(a^{\frac{1}{2}} - a^2)^2$.
19. $(a - b) \div (a^{\frac{1}{2}} - b^{\frac{1}{2}})$.
20. $(x^{\frac{2}{3}} + 2\,x^{-\frac{1}{3}} + 1)^2$.
21. $\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^{-3} \div \left(\frac{ab^{-1}}{a^{-3}b^2}\right)^5$.
22. $\left(\frac{x^{-2}y^2}{x^3y^{-2}}\right)^{-\frac{1}{2}} \times \frac{(y^3x^{-2})^{-1}}{x^3y^{-3}}$.
23. $(a^{\frac{1}{n-1}}a^{\frac{1}{n+1}})^{\frac{1}{n}}$.
24. $8^{-\frac{2}{3}} \div 256^{-\frac{1}{4}}$.
25. $\frac{m^{-1} - n^0}{m^{-2} - n^0m^{-1}} \times m^{-2}$.
26. Square $a^{\frac{1}{2}} + b^{\frac{1}{2}} - c^{\frac{1}{2}}$.
27. $12^0 + 4^{\frac{1}{2}} - 9^{-1} + (-64)^{-\frac{1}{3}} + 27^{\frac{2}{3}}$.
28. $(x^{-\frac{3}{4}} + 2\,x^{-\frac{2}{4}} + 4\,x^{-\frac{1}{4}} + 8)(x^{-\frac{1}{4}} - 2)$.

The table at the top of the next page will be found convenient for reference in simplifying and evaluating radical quantities, and evaluating the roots of quadratic equations.

TABLE OF POWERS AND SQUARE ROOTS

x	x^2	x^3	x^4	x^5	x^6	$x^{\frac{1}{2}}$	x	$x^{\frac{1}{2}}$	x	$x^{\frac{1}{2}}$
1	1	1	1	1	1	1.000	10	3.162	19	4.359
2	4	8	16	32	64	1.414	11	3.317	20	4.472
3	9	27	81	243	729	1.732	12	3.464	21	4.583
4	16	64	256	1024	4096	2.000	13	3.606	22	4.690
5	25	125	625	3125		2.236	14	3.742	23	4.796
6	36	216	1296	7776		2.449	15	3.873	24	4.899
7	49	343	2401			2.646	16	4.000	25	5.000
8	64	512	4096			2.828	17	4.123	26	5.099
9	81	729	6561			3.000	18	4.243	27	5.196

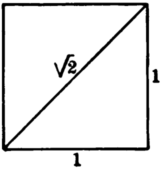
II. RADICAL EXPRESSIONS

189. Radical Quantities. An indicated root of a quantity is called a **radical quantity**, or a **radical.** Thus, $\sqrt[3]{5}$, $2^{\frac{1}{2}}$ are radical quantities. In $4\sqrt[3]{5a}$, or $4(5a)^{\frac{1}{3}}$, 4 is the **coefficient** of the radical, 3 is the **index** of the root, and $5a$ is the **radicand.**

Radical quantities may be either *rational*, as $\sqrt{25}$, $\sqrt{\frac{4}{9}}$, or *irrational*, as $\sqrt{7}$. A **rational** quantity is the **ratio** of two integral numbers. An **irrational** (or **surd**) quantity cannot be the ratio of two integral numbers.

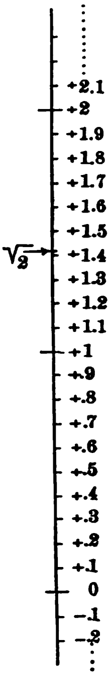
Rational quantities, as $\frac{4}{1}$, $\frac{2}{5}$, $\frac{15}{111}$, give rise, when the numerator is divided by the denominator, to either integers, finite decimals, or repeating decimals, that is, to decimals that *repeat* certain sets of figures. Thus, $\frac{5}{37} = .135135135 \cdots$; $\frac{3}{7} = .428571'428571 \cdots$. When dividing by 7, after ciphers begin to be annexed to the dividend, the possible remainders are 1, 2, 3, 4, 5, 6. When such remainders are all exhausted, the figures of the quotient must repeat in the same order.

Irrational quantities, on the other hand, never give rise to decimals like the preceding classes, but to decimals which do not end and which do not repeat the figures in regular order. An irrational number and unity have no common divisor, that is, are *incommensurable.*



In particular, it can be shown by the continued division process (§ 94) of finding the greatest common divisor that the diagonal of a square and its side, whose ratio equals $\sqrt{2}:1$, are incommensurable. By the process of § 172, we have $\sqrt{2} = 1.4142^{+}$.

The student can get a fairly good idea of the different kinds of numbers, — positive, negative, integral, rational, and irrational from the following diagram, on which the length $\sqrt{2}$, from 0 to the point indicated by the arrow, is found from the length 1 on this scale by obtaining it as the diagonal of a square whose side is 1. The diagram shows negative numbers as extending *downwards* from 0, and positive numbers *upwards* from 0; it shows also the integral numbers 1 and 2. The decimal division numbers also are examples of rationals, and $\sqrt{2}$ is an example of an irrational number.



To see the difference between rational and irrational numbers, we observe that $\sqrt{2}$ gives rise to a never ending series of decimal figures. As indicated on the diagram, we thus learn, first, that $\sqrt{2}$ lies *between* 1 and 2; next that it lies *between* 1.4 and 1.5. Now, if we subdivided the line from 1.4 to 1.5 into ten equal divisions, $\sqrt{2}$ would lie *between* 1.41 and 1.42; and if we subdivided the line from 1.41 to 1.42 into ten equal divisions, $\sqrt{2}$ would lie *between* 1.414 and 1.415; and if we subdivided the line from 1.414 to 1.415 into ten equal parts, it would lie *between* 1.4142 and 1.4143; and so on indefinitely. Thus, no matter into how many equal decimal divisions the line from 1.4 to 1.5 is divided, $\sqrt{2}$ will always lie *between* two such divisions. It can be shown that the same thing would hold true if the line were divided into any other equal divisions than tenths. The distance from 0 to a point *on* any one of the divisions described would be denoted by a rational quantity. Thus, an irrational number always locates a point different from one located by any rational number.

190. The Duplicate Notation for Radicals. As we have seen, both radical signs and fractional exponents are used to denote roots.

For example:

$$8^{\frac{1}{3}}=\sqrt[3]{8};\ a^{\frac{2}{5}}=\sqrt[5]{a^2};\ 4\,x^{\frac{1}{3}}y^{\frac{2}{3}}=4\sqrt[3]{xy^2}.$$

The reason for the existence and use of both notations is easily explained. The radical sign $\sqrt{\ }$ came into use a century before Newton and Wallis found out that fractional exponents were naturally adapted to denote roots. The use of the radical sign in the meantime had become too firmly intrenched to admit of its displacement. The radical sign notation, it may be remarked, is often shorter than the other. Thus, $(a+b)^{\frac{1}{2}}$ is more troublesome to write than $\sqrt{a+b}$, because both numerator and denominator of the fractional exponent are not written in the radical sign notation.

The use of this double notation, however, is oftentimes quite confusing to the student, and it makes the subject much more difficult than it would otherwise be.

191. Simplification of Radical Quantities. There are four kinds of **reductions** in radicals which we now proceed to investigate.

A radical is not in its simplest form:

1. *If some required root of it can be extracted.*

Thus, $(36\,x^2)^{\frac{1}{2}}=6\,x;\ 16^{\frac{1}{4}}=2;\ (27\,m^3n^6)^{\frac{1}{6}}=(3\,mn^2)^{\frac{1}{2}}.$

a. By § 13 we see that taking the square root of the cube root gives the sixth root; the square root of the square root gives the 4th root; the square root of the fourth root gives the 8th root, etc. Thus,

$$(16\,b^4)^{\frac{1}{8}}=[(16\,b^4)^{\frac{1}{4}}]^{\frac{1}{2}}=(2\,b)^{\frac{1}{2}};\ (m^3n^6)^{\frac{1}{9}}=[(m^3n^6)^{\frac{1}{3}}]^{\frac{1}{3}}=(mn^2)^{\frac{1}{3}}.$$

Type form: $$(a^m)^{\frac{1}{mn}}=a^{\frac{1}{n}}.$$

2. *If any factor of the radicand can have the desired root taken of it.*

$$(18)^{\frac{1}{2}}=(9\times 2)^{\frac{1}{2}}=3(2)^{\frac{1}{2}}.\quad (\S\ 182,\ 1.)$$

$$5(16\,x^5)^{\frac{1}{3}}=5(8\,x^3\times 2\,x^2)^{\frac{1}{3}}=10\,x(2\,x^2)^{\frac{1}{3}}.$$

If the required root of any factor of the radicand can be taken, that factor may be removed from the radicand, and its root used as a factor of the coefficient of the radical.

b. In searching for square factors, test first for 4, then for 9, then for 25, etc.

Thus, $(252)^{\frac{1}{2}} = (4 \times 9 \times 7)^{\frac{1}{2}} = 6(7)^{\frac{1}{2}}$; $450^{\frac{1}{2}} = (9 \times 25 \times 2)^{\frac{1}{2}} = 15(2)^{\frac{1}{2}}$.

Testing for cube factors, look first for 8, then for 27, etc.

Type form: $$(a^n b)^{\frac{1}{n}} = a(b)^{\frac{1}{n}}.$$

3. *If the radicand is fractional.*

To simplify, we multiply both terms of the radicand by a factor such that we can extract the desired root of the resulting denominator.

$$\left(\tfrac{3}{5}\right)^{\frac{1}{2}} = \left(\tfrac{3}{5} \times \tfrac{5}{5}\right)^{\frac{1}{2}} \text{ (§ 98) } = \left(\tfrac{1}{25} \times 15\right)^{\frac{1}{2}} = \tfrac{1}{5}(15)^{\frac{1}{2}}. \quad \text{(§ 182, 1.)}$$

$$3\left(\frac{5x}{4y}\right)^{\frac{1}{3}} = 3\left(\frac{5x}{4y} \times \frac{2y^2}{2y^2}\right)^{\frac{1}{3}} = 3\left(\frac{1}{2^3y^3} \times 10xy^2\right)^{\frac{1}{3}} = \frac{3}{2y}(10xy^2)^{\frac{1}{3}}.$$

Type form: $$\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{1}{b}(ab^{n-1})^{\frac{1}{n}}.$$

To simplify a quantity whose radicand is fractional, multiply both terms of the radicand by such a quantity as will make the denominator a perfect power of which the required root can be taken. Extract this root and write the result as a factor of the denominator of the coefficient. If possible, simplify the numerator as in 2 above.

4. *If there is an irrational quantity in its denominator.*

Simplifying such a quantity is called "rationalizing its denominator."

Thus, $$\frac{1}{6^{\frac{1}{2}}} = \frac{1}{6^{\frac{1}{2}}} \times \frac{6^{\frac{1}{2}}}{6^{\frac{1}{2}}} = \frac{6^{\frac{1}{2}}}{6}. \qquad \left(\text{Since } 6^{\frac{1}{2}} \times 6^{\frac{1}{2}} = 6^{\frac{1}{2}+\frac{1}{2}} = 6.\right)$$

$$\frac{a}{b^{\frac{2}{3}}} = \frac{a}{b^{\frac{2}{3}}} \times \frac{b^{\frac{1}{3}}}{b^{\frac{1}{3}}} = \frac{ab^{\frac{1}{3}}}{b}. \qquad \left(\text{Since } b^{\frac{2}{3}} \times b^{\frac{1}{3}} = b^{\frac{2}{3}+\frac{1}{3}} = b.\right)$$

$$\frac{4}{2+3^{\frac{1}{2}}} = \frac{4}{2+3^{\frac{1}{2}}} \times \frac{2-3^{\frac{1}{2}}}{2-3^{\frac{1}{2}}} = \frac{4(2-3^{\frac{1}{2}})}{4-3} = 4(2-3^{\frac{1}{2}}). \quad \text{(§ 65.)}$$

Verify this last result by calculating $\frac{4}{2+3^{\frac{1}{2}}}$ and the answer $(2-3^{\frac{1}{2}})$ each to three decimal places. Notice by this simpliication a long division is saved.

To rationalize a quantity with a monomial denominator, multiply oth terms by a monomial which will make the denominator rational.

To rationalize a quantity with a binomial denominator, multiply oth terms by the "conjugate surd" to the denominator; that is, the lenominator with the sign between its terms changed.

Notice that multiplying by the conjugate surd always makes he radical signs disappear. Why?

192. Exercise in Simplifying Radical Quantities. All problems iven in the radical sign notation should be immediately changed nto the fractional exponent notation, then solved, and last of all hanged back. *Before changing back into the radical sign notation, eparate quantities with fractional exponents into two factors, one vith integral exponents, and the other with proper fraction exponents.*

Thus, $5a^{\frac{5}{2}}b^{\frac{3}{2}} = 5a^2b \times a^{\frac{1}{2}}b^{\frac{1}{2}} = 5a^2b\sqrt{ab}.$

Solve as many as possible of the following mentally:

1. $(25a^3b)^{\frac{1}{2}}.$
2. $(27a^4b^3c^2)^{\frac{1}{2}}.$
3. $(108a^3b)^{\frac{1}{3}}.$
4. $(x^2y^4z^6)^{\frac{1}{4}}.$
5. $(1000)^{\frac{1}{6}}.$
6. $(1\frac{1}{16})^{\frac{1}{2}}.$
7. $\sqrt[3]{a^4x^2y^3}.$
8. $7\sqrt{396x}.$
9. $5\sqrt{726}.$
10. $\sqrt{\frac{1}{2}}.$
11. $\sqrt[3]{\frac{1}{2}}.$
12. $\sqrt[3]{\frac{5a^2}{9b}}.$
13. $\frac{3}{5}\sqrt{1\frac{2}{3}}.$
14. $\frac{a}{b}\sqrt{\frac{c^2}{d}}.$
15. $\frac{3}{45^{\frac{1}{2}}}.$
16. $\left(\frac{a^{-1}b^{-2}}{x}\right)^{\frac{1}{2}}.$
17. $\sqrt{a^2b^2+a^2c^2}.$
18. $\sqrt[3]{a^{7x}b^{3x}}.$
19. $\sqrt[5]{729a^6}.$
20. $\sqrt[6]{64a^{18}x^7}.$
21. $\sqrt{(a^2-1)(1+a)}.$

22. $\sqrt[3]{\frac{8}{9}}$. **23.** $\dfrac{3}{3^{\frac{1}{2}}+5^{\frac{1}{2}}}$. **24.** $\dfrac{1}{\sqrt{5}-\sqrt{2}}$.

25. $\sqrt[4]{\dfrac{9\,a^{-2}}{36\,x^2}}$. **26.** $\sqrt{\dfrac{a^3}{x^4}-\dfrac{a^5}{x^6}}$. **27.** $\sqrt[10]{\dfrac{32}{x^{15}y^{20}}}$.

28. Find the numerical value of $\sqrt{4\,a^2b}$ when $a=3.51$, $b=16.81$.

a. Notice the result can be had very much more quickly by first simplifying the radical. Check result by calculating radical in original form, noting the difference in the labor of calculation.

29. Calculate $\sqrt{L^2Fm}$ when $L=25.7$, $F=14$, $m=4\frac{4}{7}$.

30. Calculate area of triangle whose sides are 8, 10, and 14 (§ 89), being given from a table (p. 149) that $\sqrt{6}=2.449$.

31. Calculate the diagonal of a rectangular box from one lower corner to opposite upper corner if dimensions are 3, 4, 5 ft. and if it is known from a table (p. 149) that $\sqrt{2}=1.414$.

32. Find u from formula $u=\sqrt{v^2+2\,fs}$ when $v=3$, $f=31\frac{5}{6}$, $s=3$.

33. Find one side of a square equal to the sum of two squares whose sides are respectively 13 m. and 9 m., using the table of square roots.

34. Calculate $\sqrt{8^3\times 6^3\times 72}$, using table of square roots.

SUGGESTION. First write: $8^3=(2^3)^3=2^9$; $6^3=(2\times 3)^3=2^3\times 3^3$; $72=2^3\times 3^2$.

35. Calculate $\sqrt{\dfrac{32^2\times 4^2\times 12^5}{8^4\times 36}}$, using table.

193. Insertion of Coefficients under the Root Sign.

1. $5(3\,a)^{\frac{1}{2}}$.

SOLUTION. $5(3\,a)^{\frac{1}{2}}=25^{\frac{1}{2}}\times(3\,a)^{\frac{1}{2}}=(75\,a)^{\frac{1}{2}}$. (§ 182, 1).

2. $9(2\,m)^{\frac{1}{2}}$. **3.** $3\,x(5\,x)^{\frac{1}{3}}$. *Ans.* $(135\,x^4)^{\frac{1}{3}}$.

4. $2\,y(7\,y^3)^{\frac{1}{4}}$. **5.** $3\,x^2\sqrt{5\,xy}$. **6.** $\frac{1}{2}\sqrt[3]{4}$.

7. $\dfrac{ax}{a-x}\sqrt{\dfrac{a^2-x^2}{a^2x^2}}$. **8.** $5\,x(25\,x^2)^{\frac{1}{n}}$. **9.** $a^m\sqrt[n]{2\,a^{mn+2}}$.

194. Addition and Subtraction of Radicals. To add radical quantities they are first simplified or reduced to similar radicals.

1. Add $18^{\frac{1}{2}}$, $2\left(\frac{1}{2}\right)^{\frac{1}{2}}$, $5(64)^{\frac{1}{4}}$, and $\frac{3}{8^{\frac{1}{2}}}$.

Solution		Checking
$18^{\frac{1}{2}} = 3(2)^{\frac{1}{2}}$	(§ 191, 2.)	$\sqrt{18} = 4.24+$
$2(\frac{1}{2})^{\frac{1}{2}} = 2^{\frac{1}{2}}$	(§ 191, 3.)	$2\sqrt{.50} = 1.41+$
$5(8)^{\frac{1}{2}} = 10(2)^{\frac{1}{2}}$	(§ 191, 1, 2.)	$5\sqrt{8} = 14.14+$
$\frac{3}{8^{\frac{1}{2}}} = \frac{3}{4}(2)^{\frac{1}{2}}$	(§ 191, 4.)	$\frac{3}{\sqrt{8}} = 1.06-$.
Sum $= 14.75(2)^{\frac{1}{2}} = 20.86-$.		$20.85+$

a. By this radical simplification, which is easily and quickly performed, *four* root extractions and one long division are changed into *one* root extraction and one multiplication.

Change quantities written in the radical sign notation to the other before solving. Check some of the solutions, assigning values to the letters partly to test answers and partly for drill in root extraction. Solve as many as possible mentally.

2. $8(125)^{\frac{1}{2}} + 2(80)^{\frac{1}{2}}$.

3. $\sqrt{48\,ab^2} - b\sqrt{75\,a}$.

4. $2(\frac{1}{4})^{\frac{1}{3}} + 8\sqrt[3]{\frac{1}{32}}$.

5. $4\sqrt{128} - 5\sqrt{162} + 16\sqrt{\frac{1}{2}}$.

6. $3\sqrt{75\,a^4b} - 2\,a^2b\sqrt{\frac{12}{b}}$.

7. $3\sqrt[3]{189} - 3\sqrt[3]{875} - 7\sqrt[3]{56}$.

8. $3\,b^2(a^3c)^{\frac{1}{2}} - 2\,ab^3\left(\frac{ac}{b^2}\right)^{\frac{1}{2}}$.

9. $\frac{2}{\sqrt{2}} + \frac{3}{\sqrt{8}} - \frac{1}{\sqrt{18}}$.

10. $2\sqrt{3} - \sqrt{12} + \sqrt[4]{9}$.

11. $\sqrt[3]{\frac{1}{2}} - 2\sqrt[6]{16} + \sqrt[3]{32}$.

12. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{9}} + \sqrt{\frac{1}{8}}$.

13. $2\sqrt{\frac{5}{8}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{8}{5}}$.

14. $3\sqrt{\frac{5}{2}} + \sqrt{40} + \sqrt{\frac{2}{5}} - \frac{1}{\sqrt{10}}$.

15. $\sqrt[4]{a^2b^8} + \sqrt[3]{a^6b^2} - \sqrt[8]{a^8b^4}$.

16. $\sqrt{2\,ax^2 - 4\,ax + 2\,a} - \sqrt{2\,ax^2 + 4\,ax + 2\,a}$. (§ 31, 10)

17. From $(a-x)(a^2-x^2)^{\frac{1}{2}}$ take $a(a-x)\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$.

18. $(54\,a^{m+6}b^3)^{\frac{1}{3}} - (16\,a^{m-3}b^6)^{\frac{1}{3}} + (2\,a^{4m+9})^{\frac{1}{3}} + (2\,c^3a^m)^{\frac{1}{3}}$.

SUGGESTION. $(54\,a^{m+6}b^3)^{\frac{1}{3}} = (27\,a^6b^3 \times 2\,a^m)^{\frac{1}{3}}$; etc.

19. The areas of three lots in the form of squares lying side by side along a road are 6, 24, and 54 sq. rd., respectively. What is their total frontage on the road? Use table of square roots, page 149.

20. The areas of three squares are 32, 8, and 18 acres respectively. Find the number of rods of fence needed to inclose all of them.

21. One square lot has an area of 75 sq. rd. and another an area of 12 sq. rd. How much more fence will one need than the other?

195. Multiplication and Division of Radicals Having a Common Index. Use the fractional exponent notation in solving.

Type form: $a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$. (See § 182, 1).

1. Multiply $4\sqrt{15}$ by $3\sqrt{35}$.

SOLUTION. $4(5\times3)^{\frac{1}{2}} \times 3(5\times7)^{\frac{1}{2}} = 12(5^2\times21)^{\frac{1}{2}} = 60(21)^{\frac{1}{2}} = 60\sqrt{21}$. *Ans.*
CHECK. $15.49^{+} \times 17.75^{+} = 274.9^{+} = 275^{-}$.

2. $2(15)^{\frac{1}{2}} \times 3(5)^{\frac{1}{2}}$.
3. $8\sqrt{12} \times 3\sqrt{24}$.
4. $3(8\,a^2)^{\frac{1}{2}} \times (6\,a^3)^{\frac{1}{2}}$.
5. $\sqrt[3]{x+2} \times \sqrt[3]{x-2}$.
6. $4\sqrt{12\,a^3} \div 2\sqrt{3\,a}$.
7. $70\sqrt[3]{9} \div 7\sqrt[3]{18}$.
8. $\sqrt[4]{\frac{b}{a}} \div \sqrt[4]{\frac{a}{b}}$.
9. $\frac{3}{x}\sqrt{\frac{a^2}{x}} \times \frac{4}{3}\sqrt{\frac{x^3}{2\,a^4}}$.
10. $(6^{\frac{1}{2}} + 3^{\frac{1}{2}})(6^{\frac{1}{2}} + 5^{\frac{1}{2}})$.
11. $(9(2)^{\frac{1}{2}} - 7)(9(2)^{\frac{1}{2}} + 7)$.
12. $(7 + 3\sqrt{7})(2\sqrt{7} - 7)$.
13. $(2\sqrt{6} - \sqrt{12} - 3\sqrt{24}) \times 3\sqrt{2}$.
14. $(2\,a + 3\sqrt{x})(3\,a - 2\sqrt{x})$.
15. $(2\sqrt{8} - \sqrt{12} - 5\sqrt{18}) \div 3\sqrt{2}$.
16. $\frac{2}{5}\sqrt{\frac{4}{5}} \times \frac{1}{8}\sqrt{\frac{1}{20}}$.
17. $\frac{3}{8}\sqrt{21} \div \frac{9}{16}\sqrt{\frac{7}{20}}$.
18. $(m + n - \sqrt{mn})(\sqrt{m} + \sqrt{n})$.
19. $(9\sqrt[3]{2} - 6\sqrt[3]{6} - 3\sqrt[3]{8}) \div 3\sqrt[3]{2}$.
20. $(4\sqrt[3]{4} - 3\sqrt[3]{2})(2\sqrt[3]{6} + \sqrt[3]{9})$.
21. $(\sqrt{2} - 3\sqrt{3})^3$.
22. $(5\sqrt{8} + 6\sqrt{12} - 2\sqrt{20})(7\sqrt{2} - 3\sqrt{3} + 4\sqrt{5})$.

Rationalize the denominators of the following, evaluating the first two (see § 191, 4):

23. $\dfrac{2(5)^{\frac{1}{2}}}{5^{\frac{1}{2}}+3^{\frac{1}{2}}}$.

24. $\dfrac{15+14\sqrt{3}}{15-2\sqrt{3}}$.

25. $\dfrac{6-3\sqrt{12}}{2\sqrt{6}+\sqrt{12}}$.

26. $\dfrac{1}{\sqrt{a}+\sqrt{b}}-\dfrac{1}{\sqrt{a}-\sqrt{b}}$.

27. $\dfrac{1}{\sqrt{10}-\sqrt{2}-\sqrt{3}}$.

SUGGESTION. Make a binomial out of denominator and rationalize (§ 191, 4). Then rationalize the new denominator in a second operation.

28. $\dfrac{\sqrt{2}-\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

29. $\dfrac{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}}$.

30. If $t=2\pi\sqrt{\dfrac{l}{g}}$, and $t'=2\pi\sqrt{\dfrac{l'}{g}}$ show that $t:t'=\sqrt{l}:\sqrt{l'}$.

In the following exercises the results are to be reduced to their simplest form whenever they can be simplified. See § 191.

31. Find the surface of a cube whose volume is 60; whose volume is v.

32. Find the volume of a cube whose surface is 20; whose surface is s.

33. Ratio of area of one square to a second is $\frac{1}{3}$. Find ratio of side of first to side of second. (N. B. *The areas of similar figures are proportional to the squares of the corresponding sides.*)

34. One side of an equilateral triangle is 3. Calculate its altitude, and then find ratio of one half of one side to the altitude. Find same ratio when one side is a. Why are ratios the same?

35. Find the ratio of a side to the diagonal of a square. Let side $= a$.

36. One leg of a right triangle is one half the hypotenuse R. Find the other leg, and then the area.

37. The ratio of one leg of a right triangle to another is 2. Find the ratio of each leg to the hypotenuse.

38. The ratio of one leg of a right triangle to the hypotenuse is $\frac{1}{3}$. Find the ratio of the given leg to the other.

39. Find the area of an equilateral triangle one side of which is 6.

40. Find a side and the area of a square whose diagonal is d.

41. If a regular decagon is inscribed in a circle whose radius is R, one side of the decagon is $\frac{R(\sqrt{5}-1)}{2}$. Find the ratio of the radius to one side.

42. One leg of an isosceles triangle is 6, and the base is $\sqrt{5}$. Find the altitude and area.

43. If an equilateral △ is inscribed in a circle whose radius is R, the perpendicular distance from the center to one side is $\frac{1}{2}R$. Find ratio of a side of the triangle to the radius.

196. Multiplication and Division of Radicals which do not have a Common Index. To perform the multiplication, such quantities must be reduced to a common index. Change quantities written in radical sign notation to fractional exponent notation.

1. Multiply $18^{\frac{1}{3}}$ by $6^{\frac{1}{2}}$.

Solution.

$$18^{\frac{1}{3}} \times 6^{\frac{1}{2}} = (2 \times 3^2)^{\frac{2}{6}} \times (2 \times 3)^{\frac{3}{6}} = (2^2 \times 3^4)^{\frac{1}{6}}(2^3 \times 3^3)^{\frac{1}{6}}. \quad (\S\ 182,\ 2,\ 3.)$$

$$= (2^2 \times 3^4 \times 2^3 \times 3^3)^{\frac{1}{6}} = (2^5 \times 3^7)^{\frac{1}{6}}. \quad (\S\ 182,\ 1.)$$

$$= (3^6 \times 2^5 \times 3)^{\frac{1}{6}} = 3(2^5 \times 3)^{\frac{1}{6}} = 3(96)^{\frac{1}{6}}. \quad \textit{Ans.} \quad (\S\ 191,\ 2.)$$

2. $\sqrt[3]{6} \times \sqrt{2}$. **3.** $18^{\frac{1}{2}} \times 6^{\frac{1}{3}}$. **4.** $\sqrt[5]{64} \div 2$.

5. $\sqrt[3]{2} \times \sqrt[4]{3}$. **6.** $\sqrt[3]{10} \times \sqrt[4]{2}$. **7.** $\sqrt[m]{x^p} \div \sqrt[n]{x^q}$.

8. $\sqrt[3]{6} \times \sqrt[6]{\frac{1}{3}} \div \sqrt{8}$. **9.** $a\sqrt[m]{x} \times b\sqrt[n]{y}$. **10.** $16^{\frac{1}{3}} \times 2^{\frac{1}{2}} \times 32^{\frac{5}{6}}$.

11. $2\sqrt[3]{6} \div 6\sqrt{2}$. **12.** $\sqrt[4]{\frac{a}{b}} \times \sqrt[3]{\frac{a^2}{b^2}} \div \sqrt{\frac{b}{a}}$. **13.** $\sqrt[3]{4^2} \times \sqrt[9]{8} \times \sqrt[3]{4}$.

14. $(2\sqrt{3} - \sqrt[3]{2})(2 - \sqrt{6})$. **15.** $(\sqrt{2} - \sqrt[3]{2})^3$.

16. Arrange in order of magnitude $\sqrt{\frac{4}{3}}$, $\sqrt[3]{\frac{3}{2}}$, $\sqrt[4]{\frac{7}{4}}$.

197. Powers and Roots of Radicals. Change quantities to fractional exponent notation.

1. $[3(ab)^{\frac{1}{2}}]^3$. *Ans.* $27\,ab(ab)^{\frac{1}{2}}$.

2. $(9^{\frac{1}{3}})^2$. 3. $(3 \cdot 3^{\frac{1}{3}})^2$. 4. $(\sqrt{ax^2})^3$.

5. $(2\sqrt{\tfrac{1}{3}})^5$. 6. $\sqrt{\sqrt{a^3}}$. 7. $\sqrt[5]{\sqrt{32 \times 10^5}}$.

8. $(\sqrt[6]{3\,c^2})^3$. 9. $(\sqrt[m]{a^2b})^{m+1}$. 10. $(\sqrt{3}-\sqrt{2})^4$.

11. Simplify $\left(\frac{\sqrt{5}-3}{2}\right)^2+3\left(\frac{\sqrt{5}-3}{2}\right)+1$.

12. Simplify $2\left(\frac{7-\sqrt{17}}{4}\right)^2-7\left(\frac{7-\sqrt{17}}{4}\right)+4$.

13. Verify that $x=3+\sqrt{2}$ is a root of equation $x^2-6\,x+7=0$.

14. Verify that $\frac{1}{2}(3 \pm 2\sqrt{6})$ are roots of $4\,x^2-12\,x=15$.

15. Verify that $\frac{-11 \pm \sqrt{21}}{10}$ are roots of $5\,y^2+11\,y+5=0$.

198. The Square Root of Binomial Surds.

We have, $(\sqrt{2} \pm \sqrt{3})^2 = 2 \pm 2\sqrt{6} + 3 = 5 \pm 2\sqrt{6}$.

To extract the square root of the last expression by the method of § 171, the rational term, 5, must be separated into two parts such that the product of their square roots is $\sqrt{6}$. This can usually be done by inspection when the coefficients are small.

Extract the square root of the following by inspection:

1. $4+2\sqrt{3}$.

SOLUTION. $\sqrt{3+2\sqrt{3}+1}=\sqrt{3}+1$. (§ 171; § 184.)

2. $6-2\sqrt{5}$. 3. $9-2\sqrt{7}$. 4. $23-8\sqrt{7}$.

5. $10-\sqrt{96}$. 6. $11+\sqrt{72}$. 7. $28-5\sqrt{12}$.

When the coefficients are large, mathematicians have recourse to a formula to solve these problems. Evidently the square root of any binomial surd will have the *form* $\sqrt{x} \pm \sqrt{y}$. It is clear, also, that if the square of $\sqrt{x}+\sqrt{y}$ gives the sum, $a+\sqrt{b}$, the square of $\sqrt{x}-\sqrt{y}$ will give the difference, $a-\sqrt{b}$.

8. Let $a+\sqrt{b}$ or $a-\sqrt{b}$ denote any binomial surd, and $\sqrt{x}+\sqrt{y}$ the square root of the first and $\sqrt{x}-\sqrt{y}$ the square root of the second.

Then, (1) $\sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}}$.

(2) $\sqrt{x}-\sqrt{y}=\sqrt{a-\sqrt{b}}$

whence, $x-y=\sqrt{a^2-b}$ (Mult. Ax., multiplying (1) by (2).)

and $\underline{x+y=a}$ (By squaring eq. (1) and squaring eq. (2), adding, and dividing through by 2.)

$\therefore 2x=a+\sqrt{a^2-b}$. (Add. Ax.)

$2y=a-\sqrt{a^2-b}$. (Sub. Ax.)

whence $\sqrt{x}=\sqrt{\dfrac{a+\sqrt{a^2-b}}{2}}$. (Div. and Sq. Root Axs.)

$\sqrt{y}=\sqrt{\dfrac{a-\sqrt{a^2-b}}{2}}$. (Div. and Sq. Root Axs.)

But $\sqrt{a\pm\sqrt{b}}=\sqrt{x}\pm\sqrt{y}$. (By hypothesis.)

$$\therefore \sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

9. Extract the square root of $37-20\sqrt{3}$, using the formula just found.

Solution. $37-20\sqrt{3}=37-\sqrt{1200}$. (§ 193.)

Then, $a=37,\ b=1200$,

giving
$$\sqrt{37-\sqrt{1200}}=\sqrt{\frac{37+\sqrt{37^2-1200}}{2}}-\sqrt{\frac{37-\sqrt{37^2-1200}}{2}},$$
$$=\sqrt{\frac{37+13}{2}}-\sqrt{\frac{37-13}{2}},$$
$$=5-\sqrt{12}=5-2\sqrt{3}.$$

Proof. $(5-2\sqrt{3})^2=25-20\sqrt{3}+4\times 3=37-20\sqrt{3}$.

10. $17+\sqrt{288}$. **11.** $56+24\sqrt{5}$. **12.** $87-12\sqrt{42}$.

13. $47-4\sqrt{99}$. **14.** $56+32\sqrt{3}$. **15.** $1+\frac{2}{3}\sqrt{2}$.

16. $x-2\sqrt{x-1}$. **17.** $4\frac{1}{3}-\frac{4}{3}\sqrt{3}$. **18.** $x+xy-2x\sqrt{y}$.

III. IMAGINARIES

199. Imaginary Quantities. We have seen that such an expression as $\sqrt{-4}$ equals neither $+2$ nor -2, since the square of each of these is $+4$. The same thing is true of the square root of *any* negative number. Because such roots are neither positive nor negative ordinary numbers, they have been called *imaginaries*, though this term is somewhat misleading.

A **pure imaginary** is an even root of a negative number, as $\sqrt{-9}$, $(-6)^{\frac{1}{4}}$. The sum of a real quantity and a pure imaginary, as $2+5\sqrt{-1}$, is called a **complex number**.

The **unit** for ordinary positive numbers is $+1$; that for negative numbers is -1; and that for pure imaginaries is $\sqrt{-1}$, or $(-1)^{\frac{1}{2}}$. The $\sqrt{-1}$ unit is often denoted by the letter i.

1. By the definitions of root and power (§§ 12, 13), we have:

$(\sqrt{-1})^2=-1$, (Since $\sqrt{-1}$ means a number which

or, $([-1]^{\frac{1}{2}})^2=-1$. multiplied by itself gives -1.)

$$(\sqrt{-1})^3=(\sqrt{-1})^2(\sqrt{-1})=-1\sqrt{-1},$$

$$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=-1\times-1=1.$$

Thus, $(\sqrt{-1})^2=-1$; $(\sqrt{-1})^3=-\sqrt{-1}$; $(\sqrt{-1})^4=+1$,

or, $i^2=-1$; $i^3=-i$; $i^4=+1$.

The student is advised to solve some of the following problems as they stand, and then with i substituted for $\sqrt{-1}$, until he gets accustomed to associating the two notations.

2. Add $4+3\sqrt{-1}$ and $7+5\sqrt{-1}$. *Ans.* $11+8\sqrt{-1}$.

3. $(3+2\sqrt{-1})+(4-5\sqrt{-1})$.

4. $(-6-11\sqrt{-1})-(5-3\sqrt{-1})$.

5. $\sqrt{-4}+\sqrt{-49}-2\sqrt{-9}-3\sqrt{-25}$. *Ans.* $-12\sqrt{-1}$.

6. $\sqrt{-9a^4}+5\sqrt{-16a^4}$. (§ 191.) **7.** $3\sqrt{-25}-\sqrt{-81}$.

8. $3(-20)^{\frac{1}{2}}-(-80)^{\frac{1}{2}}+6(-45)^{\frac{1}{2}}$. *Ans.* $20(-5)^{\frac{1}{2}}$.

9. $6(-16)^{\frac{1}{2}}-[-5-(-36)^{\frac{1}{2}}]$. **10.** $(-a^2+2ab-b^2)^{\frac{1}{2}}$.

11. Multiply $6+2\sqrt{-5}$ by $3-4\sqrt{-5}$.

Solution. In all multiplications, divisions, and raising to powers of imaginary and complex numbers the safe plan is to replace $\sqrt{-1}$ by i. The reason is that following the ordinary rule for multiplication *may mislead.*

Thus, $(\sqrt{-1})^2=\sqrt{-1}\times\sqrt{-1}=\sqrt{+1}=-1$ *only*.

For, by definition of power and root $(\sqrt{-1})^2=-1$.

To change $2\sqrt{-5}$ into the i notation we write $2\sqrt{5}\sqrt{-1}=2\sqrt{5}\,i$. Then, in the answer $18\sqrt{5}\,i$ is changed back to $18\sqrt{-5}$.

For the value of i^2 see 1 of this article.

$$\begin{array}{l} 6+\ \ 2\sqrt{5}\,i \\ 3-\ \ 4\sqrt{5}\,i \\ \hline 18+\ \ 6\sqrt{5}\,i \\ \quad -24\sqrt{5}\,i-8(\sqrt{5})^2 i^2 \\ \hline 18-18\sqrt{5}\,i-8\times5\times-1=58-18\sqrt{-5}. \quad \textit{Ans.} \end{array}$$

12. $(5+\sqrt{-3})(7+4\sqrt{-3})$.

13. $(3+\sqrt{-7})(2-\sqrt{-6})$.

14. $(4+(-5)^{\frac{1}{2}})(-6-(-8)^{\frac{1}{2}})$.

15. $(6-2\sqrt{-1})^2$.

16. $\sqrt{-36}\times\sqrt{-25}\times 5$.

17. $3(-6)^{\frac{1}{2}}\times2(-4)^{\frac{1}{2}}\times(-7)^{\frac{1}{2}}$.

18. $(-3+\sqrt{-4})^3$.

19. $6(-3)^{\frac{1}{2}}\div2(-5)^{\frac{1}{2}}$.

Ans. $\frac{3}{5}(15)^{\frac{1}{2}}$.

20. $2\sqrt{-8}\div\sqrt{-2}$.

21. $2(-32)^{\frac{1}{2}}\div2(-2)^{\frac{1}{2}}$.

22. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$. (See § 191, 4.)

23. $\dfrac{\sqrt{3}-\sqrt{-5}}{\sqrt{3}-\sqrt{-2}}$.

24. Form the product of $x+a$, $x-a$, $x+a\sqrt{-1}$, and $x-a\sqrt{-1}$.

25. Form the product of $y-3$, $y-7$, $y+2+\sqrt{-7}$, and $y+2-\sqrt{-7}$.

26. Verify that both $x=-4+\sqrt{-2}$ and $x=-4-\sqrt{-2}$ satisfy the equation $x^2+8x+18=0$.

27. Verify that $x=6a\pm2a\sqrt{-1}$ satisfy $x^2-12ax+40a^2=0$.

200. Theorems concerning Equations containing Irrationals and Imaginaries.

1. *One irrational quantity cannot equal the sum of a rational quantity and another irrational quantity.*

Let $\sqrt{m}$ and $\sqrt{n}$ denote two different irrationals, m, n, and a being rational quantities, and *suppose* $\sqrt{m} = a + \sqrt{n}$.

Then, $\quad m = a^2 + 2a\sqrt{n} + n, \quad$ (Squaring Axiom.)

whence $\quad -2a\sqrt{n} = a^2 + n - m, \quad$ (Sub. Ax.)

or $\quad \sqrt{n} = \dfrac{m - n - a^2}{2a}. \quad$ (Div. Ax.)

Thus, the irrational $\sqrt{n}$ equals the rational quantity $\dfrac{m - n - a^2}{2a}$, which is contrary to § 189. Hence an irrational quantity cannot equal the sum of a rational quantity and another irrational quantity.

2. *In any equation containing rationals and irrationals, the rational part on one side equals the rational part on the other, and he irrational part on one side equals the irrational part on the other.*

GIVEN $a + \sqrt{b} = c + \sqrt{d}$, in which $\sqrt{b}$ and $\sqrt{d}$ are irrationals.

TO PROVE $\quad a = c$, and $\sqrt{b} = \sqrt{d}$.

PROOF. $\quad \sqrt{b} = c - a + \sqrt{d}. \quad$ (Sub. Ax.)

But this equation cannot exist, by 1 above, so long as $c - a$ is different from 0. Thus, $c - a$ must equal 0, whence $c = a$, and then $\sqrt{b} = \sqrt{d}$. Q.E.D.

The above theorems hold true for real and imaginary quantities as well as for rationals and irrationals.

201. Geometrical Interpretation of Imaginaries or Orthotomic Numbers. Argand's Diagram. If the word imaginary is used to describe the quantities of the preceding article with the idea that no real interpretation of them is possible, it is a misnomer.

It was shown in § 40 that -1 is a multiplier or operator which reverses the direction of the multiplicand. Now $\sqrt{-1} \times \sqrt{-1} = -1$. Thus $\sqrt{-1}$ used as a factor *twice* accomplishes as much as -1 used once. Accordingly it appears that $\sqrt{-1}$ can be regarded as a multiplier which turns a multiplicand number through an angle of 90°. Arbitrarily choosing as the positive direction for angles that opposite to the motion of the hands of